# RAMC 2022

## High School Team Solutions

*Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:*


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1. Cathy writes on the board all multiples of either 24 or 42, or both, on a whiteboard between 1 and 1000. What is the sum of the numbers on the board?

   **Answer:** $\boxed{29626}$

   **Solution:** There exist 41 multiples of 24 less than 1000, and their sum is $\frac{41\cdot(24+984)}{2} = 20664$.

   There exist 23 multiples of 42 less than 1000, and their sum is $\frac{23\cdot(42+966)}{2} = 11592$.

   We know $\text{lcm}(24, 42) = 168$. There exist 5 multiples of 168 less than 1000 with their sum being $\frac{5\cdot(168+840)}{2} = 2520$.

   Therefore, using the principle of inclusion and exclusion to prevent double counting, the final sum is $20554 + 11592 - 2520 = 29626$.

2. Let $n = 2022^{22}$. Find the number of positive divisors of $n^2$ that are less than $n$ but are not divisors of $n$.

   **Answer:** $\boxed{33396}$

   **Solution:** The prime factorization of $n$ is $n = 2^{22} \cdot 3^{22} \cdot 337^{22}$. Thus, $n^2 = 2^{44} \cdot 3^{44} \cdot 337^{22}$, so $n^2$ has $45 \cdot 45 \cdot 45 = 91225$ factors. When considering the factor pairs of $n^2$, excluding $n \cdot n$, for every pair one of the divisors must be less than $n$. There are $\frac{1}{2} \cdot (91225 - 1) = 45562$ such divisors which are less than $n$. Considering the total number of divisors of $n$, there are $23 \cdot 23 \cdot 23 - 1 = 12166$ divisors which are less than $n$. Therefore, our desired total is $45562 - 12166 = 33396$ such divisors.

3. There is a sequence of positive numbers $a_1, a_2, a_3, \cdots, a_m$ such that $a_1 = \frac{233}{9}$, $a_2 = 18$, $a_3 = 16$, and $a_m = 0$, which follows the relation

   $$a_{n+3} = a_{n+1}\left(\frac{a_n}{a_{n+2}} - 1\right) \text{ for } n = 1, 2, 3, \cdots, m.$$

   Find the value of $m$.

   **Answer:** $\boxed{15}$

   **Solution:** We can rewrite the recurrence relation as $a_{n+3}a_{n+2} = a_{n+1}a_n - a_{n+2}a_{n+1}$, which tells us that the difference between the products of the current number and the previous number and the pair one farther back is equivalent to the product of the current number and the next number. Multiplying the numbers, we see that $a_1a_2 = 466$, and $a_2a_3 = 288$. This means that $a_3a_4 = 178$, $a_4a_5 = 110$, $\cdots$, $a_{13}a_{14} = 2$, $a_{14}a_{15} = 0$. Since $a_{13}a_{14}$ is not 0, neither of those numbers are 0. Since $a_{14}a_{15} = 0$, one of $a_{14}$ and $a_{15}$ must be equal to 0, but since $a_{14}$ is not 0, then $a_{15} = 0$, so our final answer is $m = 15$.

4. Four spheres of equal radius are constructed in a cone such that they are all tangent to the base of the cone, the lateral surface of the cone, and to two other spheres. A fifth sphere with twice the radius is stacked on top of the cones such that it is tangent to all 4 spheres and to the lateral surface of the cone. What is the ratio of the height of the cone to the radius of the small spheres?

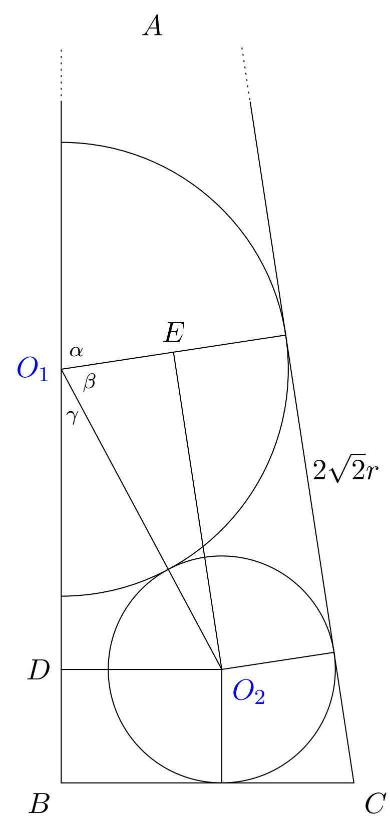
**Answer:** $\boxed{9+3\sqrt{7}}$

**Solution:** Denote the radius of the smaller sphere $r$ and the radius of the larger sphere $2r$. Take a cross section of the cone that contains the centers of the larger sphere and two opposing smaller spheres. This cross section should cut through the center of the cone. Because the centers of the smaller spheres form a square, the distance from the height of the cone to any center of a smaller sphere is $\frac{2\sqrt{2}r}{2} = \sqrt{2}r$. Consider the diagram shown, where the dotted lines continue and converge at point $A$.

Note that $EO_1 = 2r - r = r$. Thus, by the Pythagorean Theorem, $EO_2 = \sqrt{9r^2 - r^2} = 2\sqrt{2}r$ and $O_1D = \sqrt{9r^2 - 2r^2} = \sqrt{7}r$. The distance from that point to the center of the base of the cone is $r$, as it must be equal to the radius of the smaller sphere. Therefore, we only need to find the distance from the top of the cone to the center of the larger sphere.

We can do this by finding the cosine of angle $\alpha$, which is equal to the negative of the cosine of the angle $180° - \alpha$. We can find this using the sum of cosines.

$$\begin{aligned}\cos\alpha &= -(\cos\beta\cos\gamma - \sin\beta\sin\gamma)\\ &= -\left(\frac{1}{3}\cdot\frac{\sqrt{7}}{3} - \frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{2}}{3}\right)\\ &= -\left(\frac{\sqrt{7}}{9} - \frac{4}{9}\right)\\ &= \frac{4-\sqrt{7}}{9}\end{aligned}$$




Therefore, $AO_1 = \frac{2r}{\frac{4-\sqrt{7}}{9}}$ which simplifies to $8r + 2\sqrt{7}r$. This means that the height of the cone is $8r + 2\sqrt{7}r + r + \sqrt{7}r = (9+3\sqrt{7})r$, so the desired ratio is $\frac{(9+3\sqrt{7})r}{r} = 9 + 3\sqrt{7}$.

5. Balrog the frog is at the bottom of a staircase. He needs to ascend a total of 13 steps (the length of the staircase) to get to the top. His jumping power is amazing, so he can traverse any number of steps in a single jump. However, his parents get mad at him for taking the stairs too quickly if it takes less than 4 jumps to reach the top of the staircase. In how many different ways can Balrog reach the top of the staircase without his parents getting mad at him?

   **Answer:** $\boxed{4017}$

   **Solution:** Instead of thinking about the possible jump lengths Balrog can jump in a sequence, Let's think of the steps that Balrog must land on. Without the limitation of the number of jumps, he could either land or not land on any given step, with the exception of the last step, which he must land on at the end. This gives $2^{12} = 4096$ possibilities.

   However, we need to exclude the cases where Balrog takes 1, 2 or 3 jumps. There is only one possibility that requires 1 jump: Balrog jumps to the top in 1 jump.

   For the 2 jump case, Balrog can jump to any step from the $1^{\text{st}}$ and the $12^{\text{th}}$ step, then jump again to reach the top, so there are 12 cases.

   For the 3 jump case, Balrog must choose two steps between the 1st and the 12th step to land on before the last jump to the top, for a total of $\binom{12}{2} = 66$ cases.

   So, Balrog can reach the top without angering his parents in a total of $4096 - 66 - 12 - 1 = 4017$ ways.

6. Find the largest prime factor of
$$\sum_{i=3}^{50} \left[ 6\binom{i}{3} + (i-1) \right]$$

   **Answer:** $\boxed{29}$

   **Solution:** We know that $6\binom{i}{3} + i - 1 = (i)(i-1)(i-2) + (i-1) = (i^2 - 2i + 1)(i-1) = (i-1)^3$, so the sum becomes

$$\begin{aligned} \sum_{i=3}^{50} \left[ (i-1)^3 \right] &= 2^3 + 3^3 + \cdots + 50^3, \\ &= (1 + 2 + 3 + \cdots + 50)^2 - 1, \\ &= (1275)^2 - 1, \\ &= (1274)(1276). \end{aligned}$$

   By looking at each of the components of 1274 and 1276, we can find that the largest prime factor of this expression is 29.

7. A regular dodecagon (12-sided polygon) is inscribed within a circle. To the nearest integer, what percent of the area of the circle is outside of the dodecagon?

**Answer:** $\boxed{5\%}$

**Solution:** Let the radius of the circle be 1. Splitting the dodecagon into 12 triangles, we can use the Law of Sines to find that the area of each of these triangles is $\frac{1}{2} \cdot 1 \cdot 1 \cdot \sin(30^\circ) = \frac{1}{4}$. This means that the total area of the dodecagon is $12 \cdot \frac{1}{4} = 3$. The fraction of the area within the circle outside of the dodecagon is $1 - \frac{3}{\pi}$.

Let $A = 1 - \frac{3}{\pi}$. Since $3.1415 < \pi < 3.1416$, we have

$$1 - \frac{3}{3.1415} < A < 1 - \frac{3}{3.1416},$$
$$\frac{1415}{31415} < A < \frac{1416}{31416},$$
$$\frac{283}{6283} < A < \frac{59}{1309}.$$

Manually calculating each fraction to 3 decimal places, we see that $A \approx 0.045$, and rounding to the nearest integer percent yields 5%.

8. Randy the Ant walks along a coordinate plane from $(0, 0)$ to $(5, 5)$. Each step takes him one direction along the lattice. How many ways are there for Randy to get to $(5, 5)$ in 12 steps?

**Answer:** $\boxed{11088}$

**Solution:** First, notice that we can reach the point $(5, 5)$ in 10 steps as a minimum, consisting of 5 steps east and 5 steps north. That means the remaining 2 steps, regardless of where/when they are taken, must cancel each other out. We have 2 possibilities for this, either a step east and a step west or a step north and a step south. Each of these possibilities are symmetric, meaning Randy will move 6 steps in one direction, 5 in another, and 1 in a third. We have that each of these 2 paths has $\frac{12!}{6!5!} = 5544$ possibilities, so the answer is 11088.

9. Each of the following boxes in the grid below are filled with a digit from 1 to 9. Like a crossword, the number that corresponds to a given clue is the 4 digit number in the direction given by the clue starting at the grid box with the same small number as the clue's. What is the product of the upper left and the lower right digits of the grid below?

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | | | |
| 6 | | | |
| 7 | | | |

**Across:**

1-Across: Prime

5-Across: Factor of 10753344

6-Across: Digits sum to 29

7-Across: Cube

**Down:**

1-Down: Multiple of 9

2-Down: Multiple of 241

3-Down: Multiple of 24

4-Down: Multiple of 44 between 4000 and 8000

**Answer:** $\boxed{72}$

**Solution:**

| | | | |
|---|---|---|---|
| | 1 | | 7 |
| | 6 | | O |
| | 8 | | E |
| 2/1 | 7 | 4/2 | 4/8 |

Figure 1

| | | | |
|---|---|---|---|
| | 1 | | 7 |
| 2 | 6 | 6 | 7 |
| | 8 | | 8 |
| 1 | 7 | 2 | 8 |

Figure 2

| | | | |
|---|---|---|---|
| 9 | 1 | 3 | 7 |
| 2 | 6 | 6 | 7 |
| 6 | 8 | 7 | 8 |
| 1 | 7 | 2 | 8 |

Figure 3

Since 1-Across is prime, it must be odd. Since 4-down is between 4000 and 8000, we know that the first digit is between 4 and 8. The only odd digit between those 2 numbers is 5 and 7. However, if the number ended in 5, it would not be prime, so the value in the upper left corner must be 7. Additionally, 3 and 4-down are both multiples of even numbers, so the last 2 digits of 7-across must be even as well. If we list out all of the even 4 digit cubes that don't contain 0, which are $12^3 = 1728$, $14^3 = 2744$, $16^3 = 4096$, and $18^3 = 5832$ ($22^3$ is too large), the only cubes that end in 2 even numbers are 1728 and 2744, so the second digit in 7-across must be 7.

There are not many multiples of 241, less so that end in 7, so let us focus on that. For a multiple of 241 to end in a 7, it must be in the form $241 \cdot A7$, where A is any digit. Therefore, only $241 \cdot 7 = 168$, $241 \cdot 17 = 4097$, $241 \cdot 27 = 6507$, and $241 \cdot 37 = 8917$ can work. The 2nd and the 3rd possibilities contain 0, so they do not work. Since the digits of 6-across sum to 29, its second digit cannot be 1, or else the sum of digits would be too small, so 2-down must be the number 1687.

Now, using the divisibility rules of 11 and 4, we can determine the parity of all the digits in 4-down. We know that 4-down must be a multiple of 4 and a multiple of 11 since it is divisible by 44. Looking at the divisibility rules for 4, since the lower left digit can only possibly be 4 and 8, the third digit of 4-down must be even. Otherwise, the last 2 digits would not be a multiple of 4. Using the divisibility rules for 11, we see that the sum of the first and 3rd digits can at maximum and minimum be 9 and 15 respectively. The sum of the 2nd and 4th digits can, at maximum and minimum, be 5 and 17, meaning the maximum difference between the two can, at most, be 10 and -8, so by the divisibility rule for 11, the sum of the first and the third digit must equal the sum of the second and fourth digits in 4-down. Since the last 2 digits of 4-down are even, this means its second digit, which is also the last digit of 4-across, must be odd. Figure 1 is a diagram of what we know so far.

This means that 5-across is odd, so we are looking for odd factors of 10753344. It's prime factorization is $10753344 = 2^6 \cdot 3^3 \cdot 7^2 \cdot 127$. We only care about factors of $3^3 \cdot 7^2 \cdot 127 = 168021$. $3^3 \cdot 7^2 = 1323$ does not have 6 as its 3rd digit, therefore our factor needs to include 127. 127 times any single factor of 3 or 7 is not large to be 4 digits, Therefore, our possibilities are $127 \cdot 9 = 1143$, $127 \cdot 21 = 2667$, $127 \cdot 27 = 3429$, $127 \cdot 63 = 8001$, and $127 \cdot 49 = 6223$. All other possibilities are too large. Of these choices, 2667 has its second digit equal to 6, so that must be what 5-across is. By the divisibility rules of 11, this means that the 3rd digit in 4-down must be equal to its 4th digit, as the first and the second digits are equal now.

Since 24 is divisible by 8, we can use the divisibility rules for 8 to determine the last 3 digits of 3-down, so the last 3 digits of 3-down must be divisible by 8. If the last digit of 3-down were a 4, then the largest digit we can put for the second to last digit is 6, as 664 is divisible by 8, but 674, 684, and 694 are not. Since the last digit of 4-down is equal to the second to last digit of 4-down, this case would also make the second to last digit of 4-down, which is the same as the last digit of 6-across, equal to 4. The sum of digits in 6-across would, at maximum, be $9 + 8 + 6 + 4 < 29$, therefore the last digit of 3-across is 2. Filling in all the rest of the givens, our puzzle looks like Figure 2.

The only way that 3-down can be a multiple of 8 is if its 3rd digit is 3 or 7, as those are the only two numbers that will make the last 3 digits a multiple of 8. However, if it was 3, then $9+8+3+8 = 28 < 29$ is too small to make 6-across work. This means that 6-across's 3rd digit is 7. Using the clue in 6-across, this makes the first digit in 6-across equal to 29-8-7-8=6. Using the divisibility rules for 9, the digit in the upper left corner must be 9, so the product of the lower right and the upper left digits is $9 \cdot 8 = 72$.

Note that we never found the 3rd digit in 1-across, though it can be found to be equal to 3, as 9137, 9167, and 9197, the only possibilities given that 3-down is a multiple of 24. Only 9137 is prime. The final solution is given in Figure 3.

10. The polynomial $P(x) = x^{2022} + x^{2021} - 3x + 1$ has complex roots $r_1, r_2, \cdots, r_{2022}$. Find the value of $(r_1^2 + 1)(r_2^2 + 1) \cdots (r_{2022}^2 + 1)$.

**Answer:** $\boxed{4}$

**Solution:** Note that the product is equivalent to $(r_1+i)(r_1-i) \cdot (r_2+i)(r_2-i) \cdots (r_{2022}+i)(r_{2022}-i) = P(-i)P(i) = -2i \cdot 2i = 4$.