RAMC 2022

Middle School Team Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

Executive Editors

Arden Peng   Michael Huang   Golden Peng

Problem Writers

Albert Hu	Arden Peng	Felix Lu	Kevin Yang
Alexander Voskoboev	Aurora Wang	Golden Peng	Matthew Chen
Ana Milosevic	Ben Weingarten	Hans Xu	Michael Huang
Andrew Sun	Christine Song	Jason Ding	Michelle Cao
Andrew Yan	Ethan Zhang	Katherine Zhu	William Wang

Graciously Reviewed By:

Frank Lu   Lei Zhu   Leo Xu

Liya Huang   Nan Feng

1. Jake's Grocery Store has 20 identical aisles. Bob can finish cleaning the store's aisles in 1 hour by himself, and Tracy can finish in 2 hours by herself. If they both work in sync to clean 15 aisles, how many minutes does it take?

   Answer: $\boxed{30}$

   Solution: Bob takes $\frac{1}{20}$ hr $= 3$ min to clean one aisle, and Tracy takes $3 \cdot 2 = 6$ minutes. If both of them work together simultaneously, they can clean 3 aisles in 6 minutes; Bob cleans 2 while Tracy cleans 1. Therefore, that it'll take them $6 \cdot 5 = 30$ minutes to clean 15 aisles.

2. Let the function $f(x)$ reverse the order of the digits of x, and $g(x) = x + 2^x + 3^x$. Evaluate the function $f(f(g(3)) + g(4))$.

   Answer: $\boxed{481}$

   Solution: We evaluate from the inside.

$$\begin{aligned} f(f(g(3)) + g(4)) &= f(f(3 + 8 + 27) + 4 + 16 + 81) \\ &= f(f(38) + 101) \\ &= f(83 + 101) \\ &= f(184) \\ &= 481. \end{aligned}$$

3. Ana, Ben, and Catherine are running laps around the perimeter of the city's park, as shown in Diagram 1 (on the last page). In five hours, Ana can run 3 laps, Ben can run 4 and Catherine can run 5. When Damian runs for an hour, he runs an integer amount of units. He can run faster than Ana, but slower than Ben. In the time Catherine runs 6 laps, how many units will Damian run?

Answer: $\boxed{24}$

Solution: We see that one lap is formed by 3 diagonals and 2 straight lines, which makes its distance $3\sqrt{2}+2$. Ana runs $\frac{3}{5}(3\sqrt{2}+2)$ in an hour, and similarly, Ben runs $\frac{4}{5}(3\sqrt{2}+2)$. Damian runs an integer amount of units, we know that the length that he runs D is,

$$\begin{gathered}(3\sqrt{2}+2)\frac{3}{5} \leq D \leq (3\sqrt{2}+2)\frac{4}{5}\\ (3\cdot 1.41+2)\frac{3}{5} \leq D \leq (3\cdot 1.41+2)\frac{4}{5}\\ 6.23\cdot\frac{3}{5} \leq D \leq 6.23\cdot\frac{4}{5}\\ 3.74 \leq D \leq 4.98\\ D = 4\end{gathered}$$

Therefore, Damian runs 4 units every hour. Catherine runs 6 laps in $6\cdot\frac{5}{5} = 6$ hours. Therefore, Damian will run 24 units.

4. How many positive factors does 65520 have?

Answer: $\boxed{120}$

Solution: We note that

$$\begin{aligned}65520 &= 65536 - 16\\ &= 2^{16} - 2^4\\ &= 4^8 - 4^2\\ &= (4^4-4)(4^4+4)\\ &= 4^2(4^3-1)(4^3+1)\\ &= 4^2(3^2\cdot 7)(5\cdot 13)\\ &= 2^4\cdot 3^2\cdot 5\cdot 7\cdot 13.\end{aligned}$$

Therefore, the number of positive factors of 65520 is $(4+1)(2+1)(1+1)(1+1)(1+1) = 120$.

5. Let x be the shortest distance between Jolli's Bakery and Ralph's Hot Dogs, shown in Diagram 1 (on the last page), by walking only on the streets. Find the number of paths between the two points with distance x.

Answer: $\boxed{1}$

Solution: The shortest distance will use as many diagonal streets as possible, including the $\sqrt{5}$ side. The shortest distance is $5+\sqrt{2}+\sqrt{5}$, and there is only 1 path with that distance.

6. A triangle $\triangle ABC$ is circumscribed by a circle, with $\overline{AB} = 6$ and $\angle A = 120^\circ$. Extend the angle bisector from A until it intersects the circle again at point D. If $\overline{BD} = 14$, what is $\overline{AC}$?

Answer: $\boxed{10}$

Solution: We note that by the angles $\angle BCD$, $\angle CDB$, and $\angle DBC$ all correspond to a 120° arc, meaning that all the angles are 60 degrees and $\overline{BC} = \overline{CD} = \overline{BC} = 14$. Now using Law of Cosines for $\triangle ABC$, we get $6^2 + \overline{AC}^2 - 2 \cdot 6 \cdot \overline{AC} \cdot \cos(120^\circ) = 14^2$. This gives us $\overline{AC}^2 + 6\overline{AC} - 160 = 0$, or $(\overline{AC} - 10)(\overline{AC} + 16) = 0$, so $\overline{AC} = 10$.

7. Michelle has a $\frac{1}{4}$ chance of getting into each of 4 different programs and $\frac{1}{3}$ chance of getting into each of 4 other programs. If getting into each program is independent to another, what are the chances that Michelle gets into at least 1 program?

Answer: $\boxed{\frac{15}{16}}$

Solution: We know that $P(\text{Michelle gets into at least 1 program}) = 1 - P(\text{Michelle gets into no programs})$. We can evaluate that in order for Michelle to get into no programs, the probability is

$$\left(1-\frac{1}{4}\right)^4\left(1-\frac{1}{3}\right)^4 = \left(\frac{3}{4}\right)^4\left(\frac{2}{3}\right)^4 = \frac{1}{16}.$$

Therefore, the probability that Michelle gets into at least 1 program is $1 - \frac{1}{16} = \frac{15}{16}$.

8. This following table shows 3 different phone plans that Felix can purchase.

	AB&C	Horizon	V-TabLet
membership fee and duration	\$20, paid monthly	\$70, paid quarterly	\$230, paid yearly
short distance per minute	\$0.75	\$0.50	\$0.60
long distance per minute	\$1.50	\$1.25	\$1.45

Felix needs to choose a phone plan, which he will use for 8 months. How many dollars will he save by using the cheapest plan over the most expensive plan, assuming he uses 4 hours of short distance calls and 3 hours of long distance calls?

Answer: $\boxed{80}$

Solution: We can write out the amount of money this is used for each plan.

For AB&C, Felix will spend $\$20 \cdot 8 + \$0.75 \cdot 4 \cdot 60 + \$1.50 \cdot 3 \cdot 60 = \$610$.

For Horizon, he will need to pay the membership 3 times to cover for 9 months. $\$70 \cdot 3 + \$0.50 \cdot 4 \cdot 60 + \$1.25 \cdot 3 \cdot 60 = \$555$.

For V-TabLet, he will pay the yearly fee 1 time. $\$230 + \$0.60 \cdot 4 \cdot 60 + \$1.45 \cdot 3 \cdot 60 = \$635$.

As V-TabLet is his most expensive option and Horizon is his cheapest, Felix will save $\$635 - \$555 = \$80$.

9. Ralph the raccoon is moving along a triangular frame, with points e_1, e_2, and e_3 in clockwise order. Every step, Ralph moves 1 point to the left or right. When Ralph is at e_n, he has a $\frac{1}{n+1}$ chance to move to the next point clockwise. Ralph will start at e_1. What is the probability that Ralph will get to e_3 in 3 or less turns?

Answer: $\boxed{\frac{5}{6}}$

Solution:

We notice that the way that he gets it in 1 turn is by going directly to e_3, which has a probability of $1 - \frac{1}{2} = \frac{1}{2}$. The way he gets it in 2 turns is to first go to e_2, and then onto e_3, with a probability of $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. Finally, the way he gets in 3 turns is to first go to e_2, returning to e_1, and finally going to e_3, with probability $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

Therefore, the total probability that Ralph arrives at e_3 in 3 turns or less is $\frac{1}{2} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$.

10. The grid shown in Diagram 1 (on the last page) directly translates to a Cartesian plane, with each grid vertex corresponding to a lattice point, and each grid unit is length 1. Let the straight flowing river flow through the red dot on B6. The Tracy High School on D2 represents $(0, 19)$ on the Cartesian Plane. If the river has a slope of $-\frac{21}{4}$, find the number of non-negative Cartesian coordinate points strictly below the river.

Answer: $\boxed{192}$

Solution: The line for the river goes through B6, which is at $(4, 21)$ on the diagram, as the High school is at $(0, 19)$. As the river has slope $-\frac{21}{4}$, we can write:

$$y = -\frac{21}{4}x + 42.$$

By Pick's Theorem, $A = I + 1/2B - 1$ where $A =$ Area, $I =$ Interior Lattice Points, and $B =$ lattice points on the boundary.

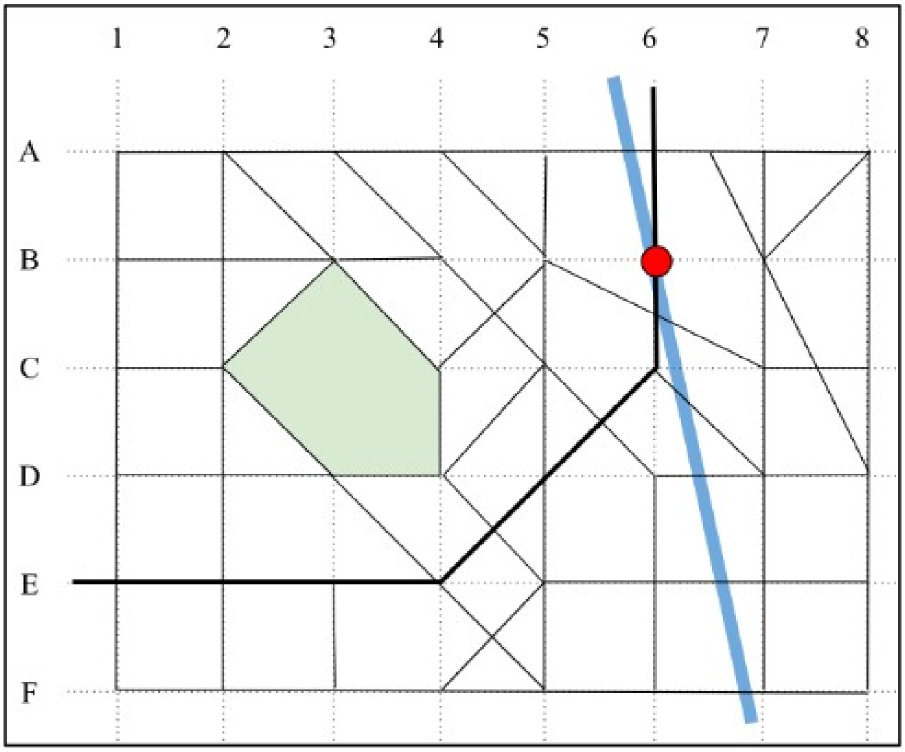
The area is the triangle with the line $y = -21x/4 + 42$, $x = 0$, and $y = 0$, is $\frac{8 \cdot 42}{2} = 168$. $B =$ All the vertices on the exterior, which include $(0, 42), (4, 21)$, and $(8, 0)$, all the values from $(0, 0)$ to $(7, 0)$, and values from $(0, 1)$ to $(0, 41)$. This is a total of $3 + 8 + 41 = 52$. Therefore,

$$168 = I + \frac{1}{2} \cdot (52) - 1,$$

which means that $I = 143$. Finally, we need to add back the ones that are on the axes but below the line, which is $41 + 8 = 49$. Therefore, there are a total of 192 points.

Diagram 1.
RMCLand Map

Scale: 1 grid length = 1 unit



All **black** lines represent roads. Dotted lines are the grid lines. Green is the city park. The straight blue line is the river.

City Directory

C8: Barbara's Barbeque and Grill
B7: Farmers Market
A4: Guo Elementary School
B5: Harry's Gym and Pool
B3: Jake's Grocery Store
A1: Jolly's Bakery
B8: Ralph's Hot Dogs
D7: Rent-a-bike
F6: Richard Middle School
E5: Ketchup Clinic *EX-PRSS* Care
D2: Tracy High School