RAMC 2022

Middle School Tiebreaker Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. How many integers between 1024 and 8192 have four distinct digits in strictly decreasing order?

   Answer: $\boxed{70}$

   Solution: Consider the numbers from 1024 to 8192. No such number can begin with 8, because the smallest number that works is 8210. No number works under 1024 either. So all numbers must have digits 0 through 7 only, where we choose 4 and arrange them highest to lowest, Therefore, there are $\binom{8}{4} = 70$ integers.

2. Adeline places into a hat one slip of paper labeled with the number "1", two slips of paper labeled "2", three slips of paper labeled "3", and four slips labeled "4". Adeline then randomly takes three slips of paper out of the hat. What is the probability that the product of the numbers on the slips of paper is divisible by 6?

   Answer: $\boxed{\frac{27}{40}}$

   Solution: In order for the product to be a multiple of 6, then Adeline's three slips must include a slip that is a multiple of 2, a slip that is a multiple of 3, and any slip. First, all of the slips may be divided into three non-overlapping categories - multiples of 2 (of which there are 6), multiples of 3 (of which there are 3), and neither (of which there is 1). To find the probability that Adeline's product is a multiple of 6, we can split the problem into three cases: Case 1: Two slips that are multiples of 2 and one slip that is a multiple of 3. There are three permutations that result in this, so the probability of this case is $3 \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{3}{10} = \frac{270}{10 \cdot 9 \cdot 8} = \frac{3}{8}$.

   Case 2: One slip that is a multiple of 2 and two slips that are multiples of 3. There are three permutations that result in this, so the probability of this case is $3 \cdot \frac{6}{10} \frac{3}{9} \cdot \frac{2}{8} = \frac{108}{10 \cdot 9 \cdot 8} = \frac{3}{20}$.

   Case 3: One slip that is a multiple of 2, one slip that is a multiple of 3, and one slip that is neither. There are six permutations that result in this, so the probability of this case is $6 \cdot \frac{6}{10} \cdot \frac{3}{9} \cdot \frac{1}{8} = \frac{108}{10 \cdot 9 \cdot 8} = \frac{3}{20}$.

   These three cases are then added together to get $\frac{15}{40} + \frac{12}{40} = \frac{27}{40}$.

3. Let $b_n = a_n + n - 1$ for all $n \geq 1$. The sequence $b_1, b_2, b_3, \cdots, b_n$ forms an arithmetic series while $a_1, a_2, a_3, \cdots, a_n$ forms a geometric sequence. If $b_{2019} = 2022$, find $a_{2022} - a_1$.

   Answer: $\boxed{0}$

   Solution: We realize that the sequence $a_1, a_2, a_3, \cdots, a_n = b_1, b_2 - 1, b_3 - 2, \cdots, b_n - (n-1)$ is an arithmetic sequence. The only sequences that are both arithmetic and geometric are sequences of a single number repeating, as proven below.

   By definition of a 4-term geometric sequence, $a_1 a_4 = a_2 a_3$, where $a_1 + (n-1)r = a_n$. We can write the equation in terms of a_1, $a_1(a_1 + 3r) = (a_1 + r)(a_1 + 2r)$. Simplifying, we have $2r^2 = 0$, so $r = 0$. Therefore, $a_1 = a_n$ for all n.

   Thus, $a_{2022} - a_1 = 0$.

4. Andrew, Damian, Hans, and Michael are counting numbers. Andrew starts with 1, Damian counts the next three numbers, Hans two, then Michael six. After this, the cycle repeats forever, but instead Andrew counts 5 numbers on all his other turns. The game ends when 10,001 is said. How many numbers did Michael say?

   Answer: $\boxed{3750}$

   Solution: Excluding the first cycle, the total amount of numbers they say in one cycle is $5+3+2+6 = 16$. We note the first cycle has 12 numbers. Therefore, we have $\frac{10001-12}{16} = 9989/16 = 624 + 5/16$ full cycles. This means, we have 624 complete cycles, which Michael says 6 in, making $624x6 = 3744$ numbers. Michael says 6 in the first 6, meaning he says $3744 + 6 = 3750$ numbers.

   Michael is last one to say the numbers in a cycle, so he won't say any of the 5 remaining numbers. Therefore, he says 3750 numbers.

5. Michelle has a $\frac{1}{4}$ chance of getting into each of 4 different programs and $\frac{1}{3}$ chance of getting into each of 4 other programs. If getting into each program is independent from another, what are the chances Michelle gets into exactly 1 program?

   Answer: $\boxed{\frac{5}{24}}$

   Solution: We can split this into two cases.

   Case 1: Michelle gets into a program that has a $\frac{1}{4}$ probability.

   The probability that Michelle gets into one of these 4 and gets rejected from all other 7 is:

$$4 \cdot \frac{1}{4} \cdot (\frac{3}{4})^3 \cdot (\frac{2}{3})^4 = \frac{1}{8}.$$

   Case 2: Michelle gets into a program that has a $\frac{1}{3}$ probability.

   The probability that Michelle gets into one of these 4 and gets rejected from all other 7 is

$$4 \cdot (\frac{3}{4})^3 \cdot \frac{1}{3} \cdot (\frac{2}{3})^3 = \frac{1}{12}.$$

   In total, the probability she gets into exactly one program is $\frac{1}{8} + \frac{1}{12} = \frac{5}{24}$.

6. Simplify $\sqrt{5^2+4^2+3^2+2^2-1^2-1^2+\sqrt{22^2+8^2+6^2+2^2}}$.

Answer: $\boxed{7+\sqrt{3}}$

Solution: We can write

$$\begin{aligned}\sqrt{5^2+4^2+3^2+2^2-1^2-1^2+\sqrt{22^2+8^2+6^2+2^2}} &= \sqrt{52+\sqrt{588}} \\ &= \sqrt{7^2+3+2\cdot 7\sqrt{3}} \\ &= \sqrt{(7+\sqrt{3})^2} \\ &= 7+\sqrt{3}.\end{aligned}$$

7. Alexander is playing a carnival game. The vendor lets him pick one of 10 identical boxes, two of which have a red ball in them. To win, he must select a box with a red ball. Once he selects a box, the vendor randomly shows him two other boxes without balls in them, and allows him to choose another box or stay with his own. What is the difference in the probabilities of winning using the optimal strategy and the worst strategy?

Answer: $\boxed{\frac{2}{35}}$

Solution: There are 2 cases: Alexander stays or he switches.

Case 1: Alexander stays. The odds don't change from the original $\frac{1}{5}$ probability if he stays.

Case 2: Alexander changes. Subcase a: The original box had a red ball in it. This means there is a $\frac{1}{7}$ chance that he chooses the correct new box, as there is only 1 red ball remaining. The probability of this subcase happening $\frac{1}{5}$, so the total probability is $\frac{1}{7}\cdot\frac{1}{5}=1/35$. Subcase b: The original box does not have a red ball in it. This means there is a $\frac{2}{7}$ chance that he chooses a correct box. There is a $1-\frac{1}{5}=\frac{4}{5}$ chance of this happening, so the total probability is $\frac{2}{7}\cdot\frac{4}{5}=\frac{8}{35}$.

This means that if he changes, there is a $\frac{1}{35}+\frac{8}{35}=\frac{9}{35}$ chance he chooses the correct box.

As $\frac{9}{35}>\frac{1}{5}$, this means that our answer is $\frac{9}{35}-\frac{1}{5}=\frac{2}{35}$.

8. If we arrange ROCHESTER into alphabetical order, we have CEEHORRST, which has no distinct vowels touching each other. We call these words "non-crossed". If we ignore English rules, the letter Y is a consonant, and the letter L can not be in any word, how many non-crossed 3-letter words are there?

Answer: $\boxed{14705}$

Solution: We can organize the number of possibilities into a few cases. Let V represent a vowel, and C represent a consonant that isn't L. We have the following possibilities: CCC, CCV, VCC, CVC, and VCV; VVC, CVV, and VVV if the vowels are the same.

configuration	# of total cases
CCC	$20^3 = 8000$
CCV	$20^2 \cdot 5 = 2000$
CVC	$20^2 \cdot 5 = 2000$
VCC	$20^2 \cdot 5 = 2000$
VCV	$20 \cdot 5^2 = 500$
VVC	$20 \cdot 5 = 100$
CVV	$20 \cdot 5 = 100$
VVV	5

Adding up all the cases, we have $8000 + 2000 \cdot 3 + 500 + 100 \cdot 2 + 5 = 14705$ total non-crossed 3-letter words.

9. Richard is a curious guy. When he is interested in a topic, he will run an experiment to learn more about it. When he runs this experiment, there is a $\frac{1}{8}$ chance of leading nowhere and giving up, where he would then give up. There is a $\frac{3}{8}$ chance of having an accident, which he would then do again, but with a $\frac{1}{4}$ chance of leading nowhere and $\frac{3}{4}$ chance of succeeding. Then, there is a $\frac{1}{2}$ chance that he succeeds, in which case he would be intrigued by his discovery and try another experiment. When Richard is interested in a topic, what's the expected number of experiments (repeated experiments count toward the total) he will run?

   Answer: $\boxed{\frac{44}{7}}$

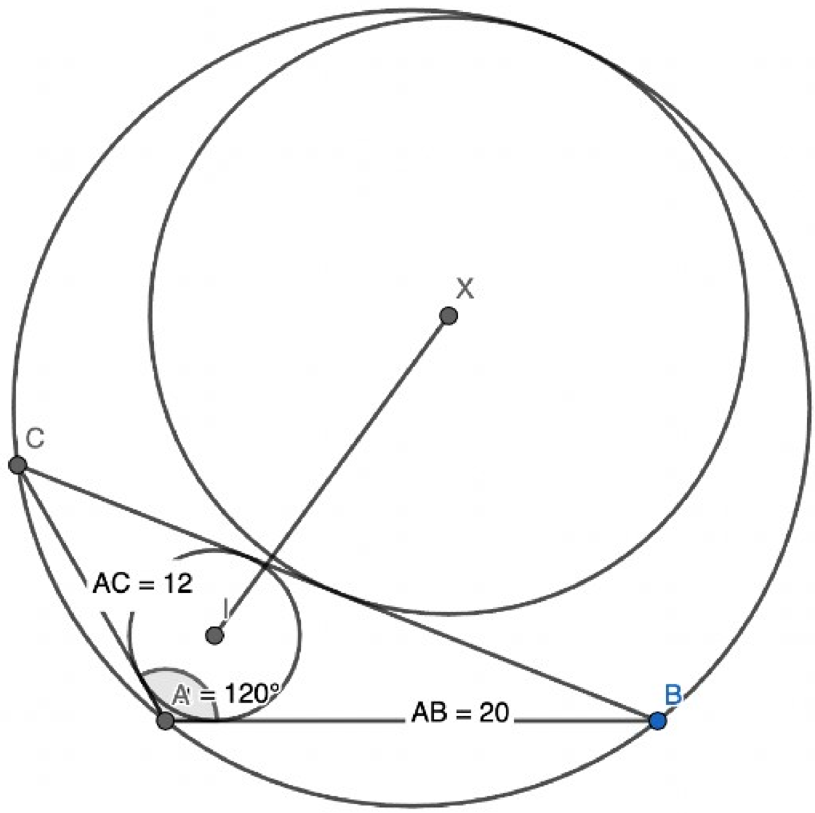
   Solution: We calculate the expected number of experiments that Richard completes, e, is

$$e = \frac{3}{8}a + \frac{1}{2}e + 1,$$

   where a is the expected number of experiments run after an accident, which is $a = \frac{3}{4}e + 1$. Substituting a into the first equation, we have

$$\begin{aligned} e &= \frac{3}{8}(\frac{3}{4}e + 1) + \frac{1}{2}e + 1 \\ e &= \frac{9}{32}e + \frac{3}{8} + \frac{1}{2}e + \frac{8}{8} \\ \frac{7}{32}e &= \frac{11}{8} \\ e &= \frac{44}{7}. \end{aligned}$$

10. A triangle $\triangle ABC$ is circumscribed by a circle, with $\overline{AB} = 12$, $\overline{AC} = 20$, and $\angle A = 120°$. Let I be the center of the incircle of $\triangle ABC$, and let X be the center of the circle tangent to the circumcircle and BC. Find $\overline{IX}^2$.



Answer: 259

Solution: Notice that because BC is a chord, the point where circle X and the circumcircle are tangent will be the midpoint of the long chord of BC, call this point N. Now also notice that the angle bisector of A extended will also intersect N. This means that $\angle BAN = 60°$, and $\angle NAC = 60°$. This in turn means that $\angle BCN = 60°$, and $\angle NBC = 60°$, because they face the same arcs on the circle as $\angle BAN$ and $\angle NAC$ respectively. This means that the triangle $\triangle NBC$ is an equilateral triangle. Since we know the values of $\overline{AB}$, $\overline{AC}$, and $\angle A$, we can find $\overline{BC}$ using the Law of Cosines, which we calculate to be 28.

Now to find $\overline{IX}$, we can use the Pythagorean Theorem. We just have to find the horizontal distance from I to X relative to BC and the vertical distance from I to X relative to BC. Both are not too hard to find with the values that we've found.

Circle X is tangent to BC at the midpoint since BC is a chord of the circle, we can call this tangency point M. Call the tangency point of the incircle to BC to be K. The horizontal distance along BC is simply KM because XM and IK are perpendicular to BC. We know $BM = \frac{1}{2}BC = 14$. BK is equal to $\frac{AB+BC-AC}{2}$ by properties of an incircle. Another way to calculate this would be to use the fact $\angle A$ is 120° so $\angle BAI$ is 60, and we can find the distance from B to the tangent of the incircle to AB, which is equal to $\overline{BK}$. Either way we get $\overline{BK} = 10$, so $\overline{KM} = 4$.

The vertical distance is just $\overline{IK} + \overline{XM}$, because they are both perpendicular to BC. Note that $\frac{IK\cdot \text{ Perimeter}}{2} =$ Area. We can find area with $\frac{1}{2} \cdot \overline{AB} \cdot \overline{AC} \cdot \sin(A) = 60\sqrt{3}$, and we know the perimeter is 60, so the inradius $\overline{IK} = 2\sqrt{3}$. XM is the radius, so it is half of the diameter MN. MN is the height of the equilateral triangle $\triangle NBC$, which has length 28. So $\overline{MN} = 14\sqrt{3}$, so $\overline{XM} = 7\sqrt{3}$. So $\overline{IK} + \overline{XM} = 9\sqrt{3}$.

As the horizontal distance is 4, and the vertical distance is $9\sqrt{3}$, so the Pythagorean Theorem gives us $\overline{IX} = \sqrt{16 + 243} = \sqrt{259}$. Therefore, $\overline{IX}^2 = 259$.