Rochester Math Circle Placement Test

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1 Introduction

This is a practice test for your personal use to determine which math circle you decide to enroll in. If you choose a level that you feel is ill-suited, please tell us, and we will make arrangements to change the level.

Set aside 90 minutes to attempt these 30 problems. Problems range from easy to difficult, although the order is not set. Please **DO NOT USE** a calculator! Good luck!

Some problems are designed to be above your level. They are meant to give you a sense of what to expect on advanced math contests. Attempt them to the best of your abilities if you have time. If you can't solve them, no worries, you will learn the methods for solving these problems and more in the program.

After you finish, please email us your answers, and solutions will be available to you once you become a member. Your score will not be used to determine your level placement and will be kept private.

2 Problems

Instructions: The word ”exact” in the questions call for an answer in simplest form. For example, instead of answering 11.8211, write instead $5 + \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} + \sqrt[5]{5}$. Good luck.

1. Calculate the sum $1 + 3 + 5 + 7 + ... + 37 + 39$.

2. Find the 32nd term of the arithmetic sequence $13, 24, 35, 46 \ldots$.

3. Equilateral triangle ABE is constructed inside unit square $ABCD$. Find exactly the area inside the square but outside the triangle.

4. Daniel flips four fair coins. If at least one lands heads, he wins the lottery. What is the probability that he wins?

5. Find exactly all values of x in the equation $\sqrt{x^2 + 8} + x = 6$.

6. Trapezoid $ABCD$ has side lengths $AB = 4$, $CD = 14$, and $BC = AD = 13$. Find exactly the area of $ABCD$.

7. 630 tokens are arranged into a triangular shape such that the first row has 1 token, the second row has 2 tokens, the third row has 3 tokens, and so on. How many tokens are there in the last row?

8. Suppose that x and y are numbers such that $\sin(x + y) = 0.7$ and $\sin(x - y) = 0.3$. Find exactly $\sin x \cdot \cos y$.

9. There are 4 red cards, 3 blue cards, and 2 white cards. All cards of the same color are identical. Five cards are chosen to be arranged in a linear fashion. How many different arrangements are possible?

10. A pyramid has a rectangular base with dimensions 12 by 30 and a height of 8. Calculate its surface area.

11. A pentagon in the coordinate plane has vertices at $(1,3)$, $(3,6)$, $(8,5)$, $(7,1)$, and $(4,0)$. Calculate its area.

12. In how many ways is it possible to seat seven people (Alex, Bill, Caroline, Dillon, Elie, Frank, Gertrude) at a round table if Alex and Bill must not sit in adjacent seats?

13. Find all possible ordered pairs (x,y), where x and y are positive integers, that satisfy $2xy+6x-5y=36$.

14. Calculate exactly the sum $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{899}+\sqrt{900}}$.

15. For a positive integer n, let $d(n)$ denote the number of divisors of n. Find all ordered pairs of primes (p,q) for which

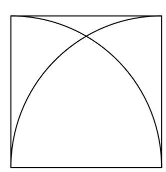
$$p\cdot d(pq)+q\cdot d(13pq)=76.$$

16. $f(x)=\frac{x^2+x}{x^4-1}+\frac{1}{x^4-x^3+x^2-x}$, calculate exactly the value $f(2)+f(3)+f(4)+\ldots+f(2017)$.

17. Isosceles triangle ABC with vertex B has a base of 18 and legs of 15. The median from point B intersects side AC at point D. The angle bisector from point A intersects side BC at point E. BD and AE intersect at point F. Find exactly the length of AF.

18. The RMC Triathlon consists of a half-mile swim, an 18-mile bicycle ride, and a 5-mile run. Richard swims, bicycles, and runs at constant rates. He runs three times as fast as he swims, and he bicycles six times as fast as he runs. Richard completes the triathlon in three and a half hours. How many minutes does he spend swimming?

19.



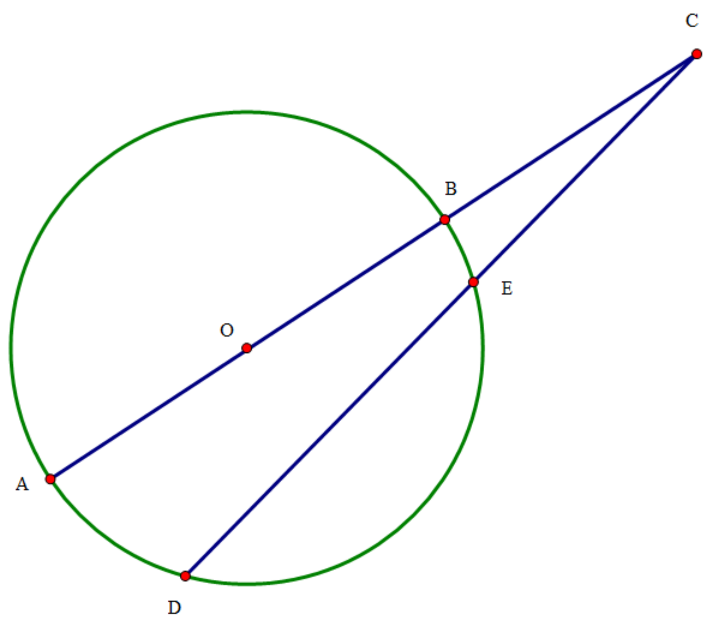
In the diagram above, two circular arcs are inscribed inside a unit square. Find exactly the area of the union of the two arcs.

20. Square $MNPQ$ has side length 2. Let A be the midpoint of PQ and B the midpoint of NP. C is on AB such that B is between A and C, and $m\angle AMB=m\angle BMC$, where $AM\neq MC$. Compute exactly the length of MC.

21. Given $a + \frac{1}{a} = 3$, find exactly:

$$\frac{a^6}{a^{12}+1}$$

22. In the figure above, line AC passes through the center of the circle O, hitting the circle at another point B. The radius of this circle is 2. Another secant line intersects the circle as shown, hitting the circle at E and D, where $CD > CE$. Given that $BC = 3$, $CE = 4$, $BD = \sqrt{7}$, and $\angle BAE = 15°$, calculate the exact area of triangle ΔAED.



23. Positive integers x and y satisfy the condition

$$\log_3(\log_{3^x}(\log_{3^y}(3^{3375}))) = 0.$$

Find the sum of all possible values of $x + y$.

24. Find the real number n such that

$$\arctan\frac{1}{4}+\arctan\frac{1}{5}+\arctan\frac{1}{6}+\arctan\frac{1}{n}=\frac{\pi}{4}.$$

25. The points $(0,0)$, $(m,8)$, and $(n,36)$ are the vertices of an equilateral triangle. Find the value of mn.

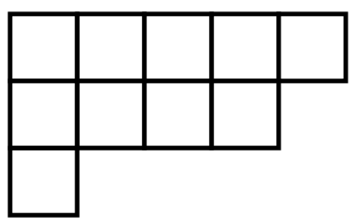
26. Find the minimum value of $\frac{16x^2\cos^2 x+9}{x\cos x}$ for $0<x<\frac{\pi}{2}$.

27. In the land of Richardtopia, the currency is measured in T coins. There are only two coins in this society: a 69 T coin, and a 343 T coin. The number "awesomeness" is an important number in this society, as it is the largest value that cannot be made by the currency. Find this number.

28. Calculate the exact value of $\sin(36)^\circ$.

29. There is one real root in the polynomial $17x^3-12x^2-6x-1=0$. Calculate it exactly.

30.



How many distinct ways can we arrange the numbers 1 to 10 into the boxes in the figure above so that, when going right, the numbers in each row are strictly increasing and, when going down rows, the numbers in each column are strictly increasing?

This is the end of the test. Please check go back and check your answers until your 90 minutes are up.