

Rochester Math Circle Placement Test (Intermediate)

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1 Introduction

This is a practice test for your personal use to determine which math circle you decide to enroll in. If you choose a level that you feel is ill-suited, please tell us, and we will make arrangements to change the level.

Set aside 60 minutes to attempt the first 15 problems. Spend another 30 minutes trying to tackle the next 15 problems. They are meant to be a challenge, to only be attempted after you are confident about the first 15. Problems range from easy to difficult, although the order is not set. Please **DO NOT USE** a calculator! Good luck!

Some problems are designed to be above your level. They are meant to give you a sense of what to expect on advanced math contests. Don't worry if you can't solve them—you will learn the methods for solving these problems in the program and as you advance to the next levels.

After you finish, please email us your answers, and solutions will be available to you once you become a member. Your score will not be used to determine your level placement and will be kept private.

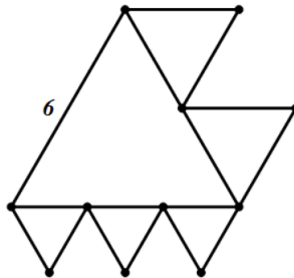
2 Problems

Instructions: The word "exact" in the questions call for an answer in simplest form. For example, instead of answering 11.8211, write instead $5 + \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} + \sqrt[5]{5}$. Good luck.

1. Find the value of x in the equation $3x + 8 = 32$.
2. What percent of 170 is 34?
3. Find the smallest integer value of x that satisfies the inequality:

$$2(x - 1) > -3(x + 5)$$

4. All of the triangles in the figure below are equilateral. Find the area of the entire figure.



5. Eight years ago, Tracy's age was one-fifth of her mom's age. Sixteen years later, her age will be half of her mom's age. How old is she this year?
6. A cylinder has a radius of 4. The ratio of the radius to the height of the cylinder is $2 : 5$. Find the volume of the cylinder.
7. Given that $a * b = 2a + b^2$, find x if $x * (2 * 3) = 201$.
8. $f(x)$ is a linear function with a positive slope. If the slope of $f(f(x)) + f(x)$ is 20, find the slope of $f(x)$.
9. What is the probability that the sum of two randomly selected factors

of 96 is odd?

10. 132 in base b is equal to 72 in base 10 . Find b .
11. Tracy wants to make a playlist of awesome music. There has to be at least one rap song and one rock song. She can choose from 5 rap songs, 3 rock songs, and 4 pop songs. Playlists with the same songs but different orders are considered the same. How many different playlists of 4 songs can she make?
12. S_1 is a geometric series, and S_2 is an arithmetic series with 10 terms. The common ratio of the terms in S_1 is equal to the opposite of the common difference of the terms in S_2 . The first term of S_1 is 3, and the fourth term is -24 . Find S_2 if its first term is 2.
13. Determine exactly the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$.
14. If the sides of a triangle have lengths 4, 5, and 6, what is the radius of the circle circumscribing the triangle?
15. Determine exactly the value of:

$$\frac{2}{\log_{12} 4} - \frac{1}{\log_{18} 4} + \frac{1}{\log_8 4}$$

3 Challenge Problems

These problems are meant to be significantly harder than the first 15. Please do not attempt until the first 15 are double-checked.

16. Rectangle $ABCD$ in the first quadrant of the coordinate plane has the given coordinates $A(1, 2)$ and $B(3, 1)$. BC has the length of $\sqrt{20}$. The area of $ABCD$ is w . The coordinates of C and D can be expressed as (m, n) and (x, y) respectively. Find $w + m + n - x - y$.
17. α is an acute angle such that $\sin \alpha = \sqrt{\frac{2x+1}{5}}$. Find exactly $\tan 2\alpha$.
18. Determine exactly $6x_3 + 3x_5$ if $x_1, x_2, x_3, x_4,$ and x_5 satisfy the system of equations below.

$$\begin{aligned}3x_1 + x_2 + x_3 + x_4 + x_5 &= 4 \\x_1 + 3x_2 + x_3 + x_4 + x_5 &= 15\end{aligned}$$

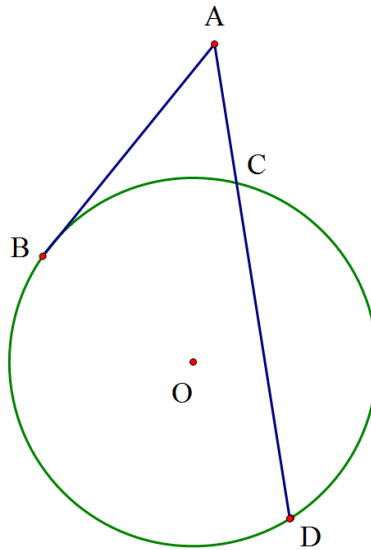
$$x_1 + x_2 + 3x_3 + x_4 + x_5 = 22$$

$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 54$$

$$x_1 + x_2 + x_3 + x_4 + 3x_5 = 72$$

19. What is the remainder when $2^0 + 2^1 + 2^2 + \dots + 2^{69421}$ is divided by 7?

20.



In the figure above, segment AB is tangent to circle O at B . Given that that $OB = 5$, $BA = 6$, and $AC = 4$, find exactly the area of the quadrilateral $ABOC$.

21. Find the least positive integer greater than $(\sqrt{5} + \sqrt{3})^6$.

22. Find exactly the largest possible distance between two points, one on the sphere of radius 15 with center $(-7, -12, 12)$ and the other on the sphere of radius 55 with center $(18, 12, -16)$.

23. Richard and Tracy sprint for 60 minutes on a circular track. Richard runs clockwise at 200 meters per minute and uses the inner lane of the track that has radius 50 meters. Tracy runs counterclockwise at 240 m/min and uses the outermost lane with radius of 60 meters. She starts on the same radial line as Richard. After they start, how many times do they pass each other?

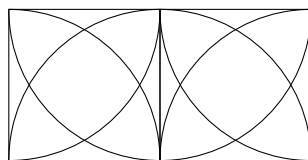
24. An unfair eight-sided die with faces numbered 1 – 8 has the properties that when rolled, the probability of obtaining face F is less than $\frac{1}{8}$, the

probability of obtaining the face opposite of F is greater than $\frac{1}{8}$, and the probability of obtaining the six remaining faces is $\frac{1}{8}$. The sum of numbers on opposite faces is 9. When two such dies are rolled, the probability of obtaining a sum of 9 is $\frac{79}{512}$. What is the probability of obtaining face F ?

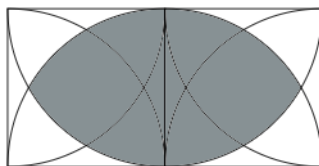
25. Triangle ABC has side lengths $AC = 9$ and $BC = 8$. A cevian from B intersects AC at E . The median from A intersects BC at D and has a length of 10. AD and BE intersect at F , and $BF : FE = 3 : 1$. Find exactly $\cos(\angle AFB)$.

26. For all positive integers m , let $f(m) = \log_{2261} m^2$. Let $X = f(7) + f(17) + f(19)$. Find exactly the range of values X can take.

27.



Two identical squares are put together in the diagram above. Each square has side length 2, with four quarter circles inscribed within the square (each quarter circle is centered at each of the vertices of the square).

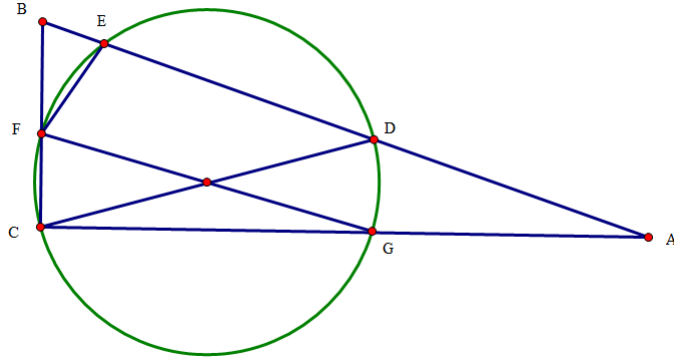


A portion of the figure is now shaded. Calculate the area of the shaded region.

28. How many sequences of 10 characters consist entirely of R 's and T 's and that have the property that every consecutive run of R 's has even length, and every consecutive run of T 's has an odd length? For example, sequences RR , T , and $RRTRR$ are valid, but $TTRT$ is not.

29. Compute exactly the area of the polygon whose vertices are the solutions in the complex plane to the polynomial $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

30.



In the figure above, CD is the diameter of the circle. Triangle ABC has right angle at C . Given that D is the midpoint of AE , $BC = 4\sqrt{5}$, $AC = 2$, and E, F, G all lie on the circle, is $m\angle B = m\angle EFG$? In addition, calculate exactly the length of the diameter of the circle.

This is the end of the test.