2nd Rochester Area Math Competition 2020

12 September 2020

High School Individual

1. In a racing game, players can choose between a kart with 4 green wheels and a bike with 2 red wheels. On the game map of Enchanted Valley, there are a total of 10 vehicles and 26 wheels. How many bicycles are there in Enchanted Valley?

2. For a real number x, the positive side lengths of an isosceles triangle are $22 - x^2$, $-x^2 + 6x + 10$, and $-x^2 + 12x - 14$. What is the largest possible perimeter of the triangle?

3. A cube has vertices placed at $(2, 3, 4)$ and $(3, 3, 4)$. The product of all possible positive volumes of the cube can be expressed in the form $\frac{a}{b}\sqrt{c}$, where a, b, and c are positive integers; a and b are relatively prime; and c is not divisible by the square of any prime. Compute $a + b + c$.

4. The following expression can be expressed concisely as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Compute $a + b$.

$$\sqrt{2 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots}}}}$$

5. 10 students are on a field trip to the grand Canyon. They divide into two group of 5, with one group heading to Havasu Falls and another group heading to the Rim Trail. Suppose that two of the students, Chris and Tiffany, insist on being in the same group, while a different pair of students, John and Cynthia, insist on being in different groups. How many distinct ways are there to assign the students to locations?

6. A function f satisfies the equation $2f(x) - \frac{1}{x^2}f(-\frac{1}{x}) = 3$ for all real values of x outside of 0. The value of $f(3)$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

7. Find the smallest integer $n > 1$ such that n^3 has more than 64 divisors.

8. A magician is standing at $(0, 0)$ in the xy coordinate plane. From any point (x, y), she can use a spell to travel to $(x, y + 1)$, $(x + 1, y + 1)$, or to travel to $(x + 1, y)$. Suppose that she journeys from $(0, 0)$ to $(4, 5)$ using a sequence of spells. If she selects one of these sequences at random, the probability that she will pass by a crater at $(2, 3)$ can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $a + b$.

9. The vertices of a triangle lie on the graph of $f(x) = 3^x$ and their x-coordinates form an arithmetic sequence of positive integers. If the area of the triangle is 1728, compute the x-coordinate of the rightmost vertex.

10. Two nonzero numbers x and y are chosen independently and at random in the interval $[-5, 5]$. The probability that the two numbers satisfy the inequality can be expressed in the form $\frac{a\pi+b}{c}$, where a, b, and c are positive integers and $\gcd(a, b, c) = 1$, where $\gcd(a, b, c)$ denotes the greatest common divisor of a, b, c. Compute $a + b + c$.

$$\frac{x^2 + y^2}{|x| + |y|} \leq 4.$$

11. A cube with side length 4 has its surface painted green and its corners are trimmed off so that each of the 6 original faces becomes a regular octagon. The unpainted faces on each corner of the new solid are then painted in yellow. The new solid is rested so that one of its yellow faces is touching the ground. The height of the new solid can be expressed in the form $\frac{\sqrt{a}+\sqrt{b}}{3}$, where a and b are positive integers. Compute $a + b$.

12. Line segment $\overline{AB}$ is placed on the xy coordinate plane with A located at $(2, 2)$ and B located at $(4, 3)$. $\overline{AB}$ is then rotated n degrees counterclockwise, where $0 < n < 180$, about the origin so that A now lies on the y-axis. The sum of the new x and y coordinates for B can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Find $a + b$.

13. In a video game, a row of blocks B_1, B_2, B_3,..., B_{10} are located in a lagoon. Two players are currently standing on block B_1. From the kth block B_k, for $1 \leq k \leq 10$, Player 1 can bounce to either B_{k+1},B_{k+2}, or B_{k+3}. Player 2 is less experienced in the game and he can bounce to either B_{k+1} or B_{k+2} from B_k but he cannot reach B_{k+3} with one bounce. Let M_i be the number of distinct sequences of bounces that Player i can use to move from block B_1 to the final block B_{10}. Compute $M_1 - M_2$.

14. Integers x, y and the real number a satisfy the following two equations.

$$(\log_{17} x)^2 + (\log_{13} y)^2 + 2a^2 = 2a(\log_{17} x + \log_{13} y).$$

$$\log_{221}(x^2 + y^2 - 13) = a.$$

If x and y are both greater than 1, compute the exact value of $x + y$.

15. For a real number k greater than 1, define the function $f(k)$ to be

$$f(k) = \arctan\left(\frac{1}{k}\right) - \ln\left(\left|1 - \frac{1}{k}\right|\right)$$

Note that $\ln()$ is the natural algorithm with base e. Find the positive value n that satisfies

$$f(3) + f(7) + f(8) = \frac{\pi}{4} - \ln\left(\frac{n-1}{2n}\right) - f(n).$$