# RAMC 2022

## High School Individual Round

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- **SCORING:** The first 10 questions are worth 1 point each, and the last 5 questions are worth 2 points each.
- This round contains 15 questions to be solved in 45 minutes. Problems towards the end tend to be more difficult than problems toward the beginning.
- No computational aids are permitted other than scratch paper, graph paper, and a pen/pencil. No calculators of any kind are allowed.
- All answers must be in a reasonably simplified form.
- Fill out your information, and sign/initial the honor code on the answer sheet provided.
- If you believe there is an error on the test, submit a challenge to the proctors. Please include your name, level (Elem I/II, MS, HS), and explanation of the problem and your solution.

Do not flip the page until the proctor begins the round!

1. Find the ordered triple, $(x, y, z)$, which satisfies the following system of equations:

$$\frac{1}{y} + \frac{1}{xz} = 3, \qquad \frac{1}{x} + \frac{1}{y} = 2, \qquad \frac{1}{x} + \frac{1}{z} = 3.$$

2. Find the sum of the unique prime factors of 72899.

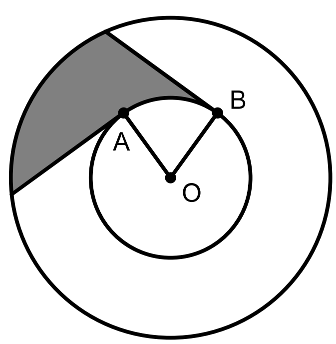
3. A florist named Kelly plants a rectangular garden full of violets with positive integer dimensions $x$ and $y$ meters. Kelly also wants to surround this garden with a rectangular moat with a width of 1 meter on all sides, but she wants the moat's area to be one-third the area of the garden. How many possible ordered pairs, $(x, y)$, exist for the dimensions of this garden?

4. Aether and Lumine are planning to meet at a cafe. However, they did not tell each other the exact time they would meet up. Each person will show up at a random time, chosen uniformly, between 4 and 5 PM. Aether will wait at the cafe for Lumine to show up for 30 minutes. Lumine, who is more impatient, is only willing to wait 10 minutes. What is the probability that the two meet up?

5. Find the sum of all positive solutions $x$, for $0 \leq x < 2\pi$, that satisfy the equation $\dfrac{\sin(x) + \cos(x)}{\sin(x) - \cos(x)} = \tan(x)$.

6. Pyxis, the point, is wandering around the $xy$-plane. He starts at the origin facing the positive $x$ direction. For his first step, he moves 2 units forward and then rotates $90^\circ$ counterclockwise. For every future step, he moves forward two-thirds the length of the previous step and then rotates $90^\circ$ counterclockwise. As he takes more and more steps, he will get arbitrarily closer to a single point. What is the sum of the $x$ and $y$ coordinates of this point?

7. Two concentric circles of radii 1 and 2 are centered about $O$. Points $A$ and $B$ lie on the smaller circle such that $\angle AOB = 72^\circ$. Tangents at $A$ and $B$ are drawn as shown below to enclose the shaded region. Compute the area of the shaded region.



8. Suppose that $k$ is such that
$$\sum_{x=1}^{100} \log\left(\frac{x^2+9x+18}{x^2+9x+20}\right) = \log k$$
Find $k$.

9. Find the largest integer $n \leq 2022$ such that $3^n + 2^n$ is a multiple of 7.

10. Triangle $ABC$ has side lengths $AB = 13$, $BC = 14$, and $AC = 15$ with centroid $G$. Two circles tangent to side $BC$ are constructed: one passes through $A$ and $B$; the other passes through $A$ and $C$. The two circles intersect at some point $P \neq A$. Find the length of segment $PG$.

11. How many sequences of "X"s and "O"s of length 14 do not contain any consecutive "X"s and do not have a run of 3 consecutive "O"s?

12. Let $P(x)$ be a monic (leading coefficient of 1) polynomial with minimal degree such that $P(n) = (n^2-1)^{-1}$ for $n = 2, 3, \cdots, 2022$. Compute the largest prime divisor of $P(0) + 1$.

13. Amber, Bennett, Childe, Diona, and Eula are trying to form 3 committees, committee A, committee B, and committee C. Any person in committee B or committee C must also be in committee A. However, no one can be in both committee B and committee C. None of the committees can be empty. In how many ways can these 5 people form committees?

14. On the Cartesian plane, point $A$ is at $(1, 5)$, point $B$ is on the $y$-axis and point $C$ is on the line $y = x$. What is the minimum possible perimeter of triangle $ABC$?

15. Let $\varphi(n)$ denote the number of positive integers at most $n$ which are relatively prime to $n$. Denote, by $S$, the sum
$$S = \sum_{i=1}^{2021^{11}} \varphi(\gcd(i, 2021^{11})).$$
Compute the number of positive divisors of $S$.