RAMC 2022

High School Individual Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. Find the ordered triple, (x, y, z), which satisfies the following system of equations:

$$\frac{1}{y}+\frac{1}{xz}=3, \qquad \frac{1}{x}+\frac{1}{y}=2, \qquad \frac{1}{x}+\frac{1}{z}=3.$$

Answer: $\boxed{(1,1,\frac{1}{2})}$

Solution: Let $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$. Substituting for these variables yields the equations: $b+ac=3$, $a+b=2$, and $a+c=3$. Adding all of them together, we have $2(a+b)+c(1+a)=8$, and since $a+b=2 \implies 2(a+b)=4$, then $c+ac=4$. Substituting $c=\frac{4}{1+a}$ into $a+c=3$, we get the quadratic $a^2-2a+1=0$. Therefore, $a=1$, $c=2$, $b=1$, and the corresponding triple $(x,y,z)=(1,1,\frac{1}{2})$.

2. Find the sum of the unique prime factors of 72899.

Answer: $\boxed{540}$

Solution: Note that $72899=270^2-1=(270+1)(270-1)=269\cdot 271$, which are both prime. Thus, the sum of it's unique prime factors is $269+271=540$.

3. A florist named Kelly plants a rectangular garden full of violets with positive integer dimensions x and y meters. Kelly also wants to surround this garden with a rectangular moat with a width of 1 meter on all sides, but she wants the moat's area to be one-third the area of the garden. How many possible ordered pairs, (x, y), exist for the dimensions of this garden?

Answer: $\boxed{10}$

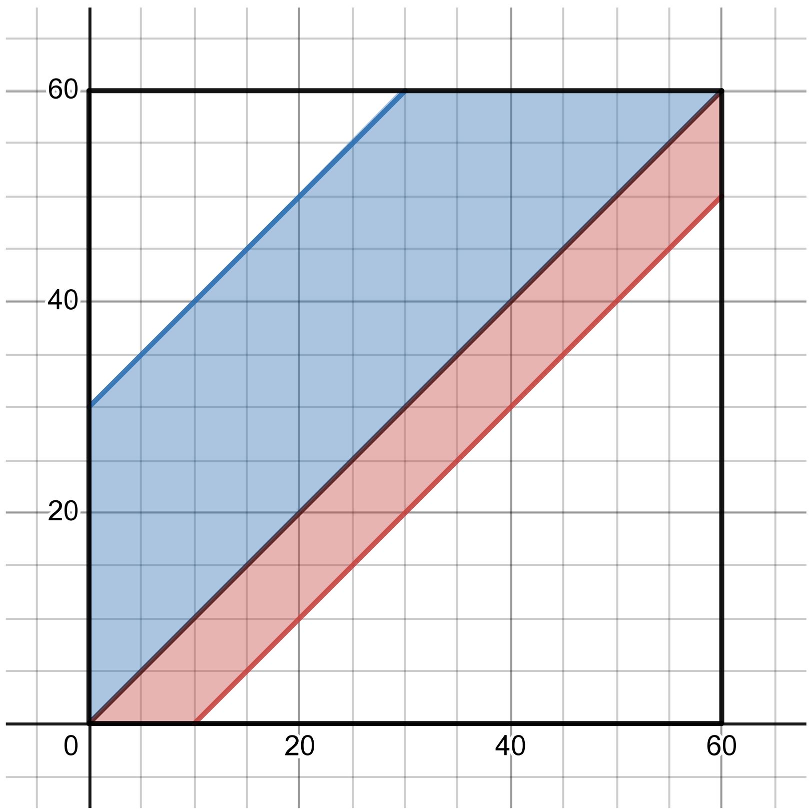
Solution: Setting the area of the rectangular perimeter to one-third of the garden's area, we get $(x+2)(y+2)-xy=\frac{1}{3}xy$. Expanding the equation and using Simon's Favorite Factoring Trick, we get $xy-6x-6y=12 \implies (x-6)(y-6)-36=12 \implies (x-6)(y-6)=48=2^4\cdot 3$. Since 48 has 10 factors and thus 5 factor pairs, and each pair accounts for 2 solutions, there exist 10 total possible ordered pairs for the dimensions of the garden.

4. Aether and Lumine are planning to meet at a cafe. However, they did not tell each other the exact time they would meet up. Each person will show up at a random time, chosen uniformly, between 4 and 5 PM. Aether will wait at the cafe for Lumine to show up for 30 minutes. Lumine, who is more impatient, is only willing to wait 10 minutes. What is the probability that the two meet up?

   Answer: $\boxed{\frac{19}{36}}$

   Solution: Instead of thinking about this problem combinatorially, consider a coordinate system. Let the x coordinate be the number of minutes past 4 that Lumine shows up, and let the y coordinate be the number of minutes past 4 that Aether shows up.

   If Aether shows up before Lumine, which occurs above the line $x = y$, then Aether and Lumine will meet if Lumine comes within 30 minutes that Aether comes, which occurs below the line $x + 30 = y$. If Lumine shows up before Aether, then our pair of coordinates needs to be above the line $x - 10 = y$. This produces the graph shown.



   In order to find the probability that they successfully meet, we just need to find the ratio of the area of the shaded region to the entire time frame. The area of the shaded region is $60 \cdot 60 - \frac{1}{2} \cdot 30 \cdot 30 - \frac{1}{2} \cdot 50 \cdot 50 = 1900$, and the whole square has area $60^2 = 3600$. Thus, the desired probability is $\frac{19}{36}$.

5. Find the sum of all positive solutions x, for $0 \leq x < 2\pi$, that satisfy the equation $\frac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)} = \tan(x)$.

Answer: $\boxed{\frac{9\pi}{2}}$

Solution: Firstly,

$$\frac{\sin(x)+\cos(x)}{\sin(x)-\cos(x)} = \frac{\sin(x)}{\cos(x)}.$$

Multiplying out the numerator and denominators, we get

$$\sin(x)\cos(x) + \cos^2(x) = \sin^2(x) - \sin(x)\cos(x),$$
$$\cos^2(x) - \sin^2(x) = -2\sin(x)\cos(x).$$

Applying double angle identities,

$$\cos(2x) = -\sin(2x).$$

Thus, we see that all possible values of $2x$ are in the form $2\pi n + \frac{3\pi}{4}$ or $2\pi n + \frac{7\pi}{4}$ for $n \in \mathbb{N}_0$.

For $0 \leq x < 2\pi$, the possible values of x are: $\frac{3\pi}{8}$, $\frac{7\pi}{8}$, $\frac{11\pi}{8}$, and $\frac{15\pi}{8}$, the sum of which is $\frac{9\pi}{2}$.

6. Pyxis, the point, is wandering around the xy-plane. He starts at the origin facing the positive x direction. For his first step, he moves 2 units forward and then rotates 90° counterclockwise. For every future step, he moves forward two-thirds the length of the previous step and then rotates 90° counterclockwise. As he takes more and more steps, he will get arbitrarily closer to a single point. What is the sum of the x and y coordinates of this point?

Answer: $\boxed{\frac{30}{13}}$

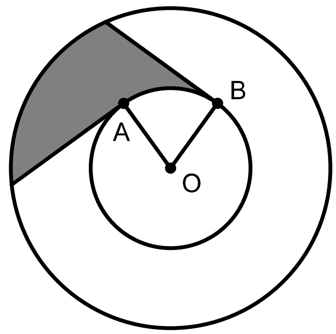
Solution: Since Pyxis rotates 90° counterclockwise every step, he travels in all directions in the following order: $+x$, $+y$, $-x$, $-y$. This means that every other step, he travels in the same orientation, either x or y.

Pyxis travels 2 units in the x-direction for the first step, so he travels $-2 \cdot \left(\frac{2}{3}\right)^2$ units in the x-direction for the third step. Thus, all the x-direction movements will follow an infinite geometric sequence with first term 2 and common ratio $-\left(\frac{2}{3}\right)^2$. So, the approached x-coordinate he approaches is given by the infinite sum, $\frac{2}{1+\left(\frac{2}{3}\right)^2} = \frac{18}{13}$.

Similarly, all the y-direction movements follow an infinite geometric sequence with first term $\frac{4}{3}$ and common ratio $-\left(\frac{2}{3}\right)^2$. The approached y-coordinate is $\frac{\frac{4}{3}}{1+\left(\frac{2}{3}\right)^2} = \frac{12}{13}$.

Thus, the sum of the x and y coordinates is $\frac{30}{13}$.

7. Two concentric circles of radii 1 and 2 are centered about O. Points A and B lie on the smaller circle such that $\angle AOB = 72^\circ$. Tangents at A and B are drawn as shown below to enclose the shaded region. Compute the area of the shaded region.



Answer: $\boxed{\frac{3\pi}{5}}$

Solution: Note that rotating the shaded area by 72° five times covers entirely the annulus between the two circles. Therefore, the shaded area is one-fifth the area of the annulus, or $\frac{1}{5}(4\pi - \pi) = \frac{3\pi}{5}$

8. Suppose that k is such that

$$\sum_{x=1}^{100} \log\left(\frac{x^2 + 9x + 18}{x^2 + 9x + 20}\right) = \log k$$

Find k.

Answer: $\boxed{\frac{53}{78}}$

Solution: Factoring yields

$$\sum_{x=1}^{100} \log\left(\frac{(x+6)(x+3)}{(x+4)(x+5)}\right) = \log k.$$

Using logarithm properties, we can convert to a product:

$$\log\left(\prod_{x=1}^{100} \frac{(x+6)(x+3)}{(x+4)(x+5)}\right) = \log k.$$

So,

$$k = \prod_{x=1}^{100} \frac{(x+6)(x+3)}{(x+4)(x+5)}.$$

Translating into factorials yields

$$k = \frac{\frac{106!}{6!} \cdot \frac{103!}{3!}}{\frac{104!}{4!} \cdot \frac{105!}{5!}} = \frac{106! \cdot 103! \cdot 4! \cdot 5!}{6! \cdot 3! \cdot 104! \cdot 105!} = \frac{106 \cdot 4}{104 \cdot 6} = \frac{53}{78}.$$

9. Find the largest integer $n \leq 2022$ such that $3^n + 2^n$ is a multiple of 7.

Answer: $\boxed{2019}$

Solution: Rearranging, the condition is equivalent to:

$$\begin{aligned}3^n + 2^n &\equiv 0 \pmod 7,\\ 3^n &\equiv -2^n \pmod 7,\\ \left(\frac{3}{2}\right)^n &\equiv -1 \pmod 7,\\ 5^n &\equiv -1 \pmod 7.\end{aligned}$$

Note that $5^3 \equiv -1 \pmod 7$ and $5^6 \equiv 1 \pmod 7$, so n must be of the form $3 + 6k$ for some integer k (i.e. an odd multiple of 3). The largest odd multiple of 3 that is at most 2022 is 2019.

10. Triangle ABC has side lengths $AB = 13$, $BC = 14$, and $AC = 15$ with centroid G. Two circles tangent to side BC are constructed: one passes through A and B; the other passes through A and C. The two circles intersect at some point $P \neq A$. Find the length of segment PG.

Answer: $\boxed{\frac{\sqrt{37}}{222}}$

Solution: Extend AP to meet BC at M. Note from power of a point on P that $MP \cdot MA = MB^2$ and $MP \cdot MA = MC^2$. Therefore, $MB \cong MC$ so AM is a median and A, G, M are collinear.

From Stewart's Theorem,

$$\begin{aligned}AM^2 &= \frac{1}{2}\left(AB^2 + AC^2 - 2 \cdot BM^2\right) = 148,\\ AM &= 2\sqrt{37}.\end{aligned}$$

Then, by power of a point, $MP = \frac{49}{2\sqrt{37}} = \frac{49\sqrt{37}}{74}$, and $MG = \frac{2\sqrt{37}}{3}$, so $PG = MG - MP = \frac{\sqrt{37}}{222}$.

11. How many sequences of "X"s and "O"s of length 14 do not contain any consecutive "X"s and do not have a run of 3 consecutive "O"s?

Answer: $\boxed{86}$

Solution: Let A, B, and C sequences be sequences that end with "OX", "XO", or "OO", respectively, and let A_k, B_k, and C_k denote the number of such sequences with length $k \geq 2$. Note that these are the only ways a sequence that follows the restrictions in the statement can end.

Consider the cases when we try to tack on another "X" or "O" onto the end of these three types of sequences. An A sequence can only have an "O" tacked on, after which it becomes a B sequence. A C sequence can only have an "X" tacked on, after which it becomes a A sequence. A B sequence can have either an "O" or an "X" tacked on, after which it becomes either an A or a C sequence, respectively.

We can now construct a system of recurrences that model the situation: $A_k = B_{k-1} + C_{k-1}$, $B_k = A_{k-1}$, $C_k = B_{k-1}$.

Rearranging, we find that $A_k = A_{k-2} + A_{k-3}$, and that $A_k + B_k + C_k = A_k + A_{k-1} + A_{k-2}$. So, to find the total number of sequences with length 14, we need to find the sum $A_{14} + A_{13} + A_{12}$.

We know that $A_2 = B_2 = C_2 = 1$, so $A_3 = B_2 + C_2 = 2$, and $A_4 = B_3 + C_3 = A_2 + B_2 = 2$. Using that $A_k = A_{k-2} + A_{k-3}$, the pattern continues:

k	2	3	4	5	6	7	8	9	10	11	12	13	14
A_k	1	2	2	3	4	5	7	9	12	16	21	28	37

Thus, the answer is $21 + 28 + 37 = 86$.

12. Let $P(x)$ be a monic (leading coefficient of 1) polynomial with minimal degree such that $P(n) = (n^2 - 1)^{-1}$ for $n = 2, 3, \cdots, 2022$. Compute the largest prime divisor of $P(0) + 1$.

Answer: $\boxed{2017}$

Solution: Consider the polynomial $Q(x) = (x^2 - 1)P(x) - 1$. The given conditions imply that $Q(x)$ is a degree 2021 polynomial with roots $2, \cdots, 2022$. Moreover, since P is monic, so is Q. Hence, $Q(x) = (x-2)(x-3)\cdots(x-2022)$. Therefore,

$$(x-2)(x-3)\cdots(x-2022) = (x^2 - 1)P(x) - 1.$$

Hence, $P(0) = 2022! + 1$ and so the largest prime divisor of $P(0) - 1$ is simply the largest prime at most 2022, which is 2017.

13. Amber, Bennett, Childe, Diona, and Eula are trying to form 3 committees, committee A, committee B, and committee C. Any person in committee B or committee C must also be in committee A. However, no one can be in both committee B and committee C. None of the committees can be empty. In how many ways can these 5 people form committees?

    Answer: $\boxed{570}$

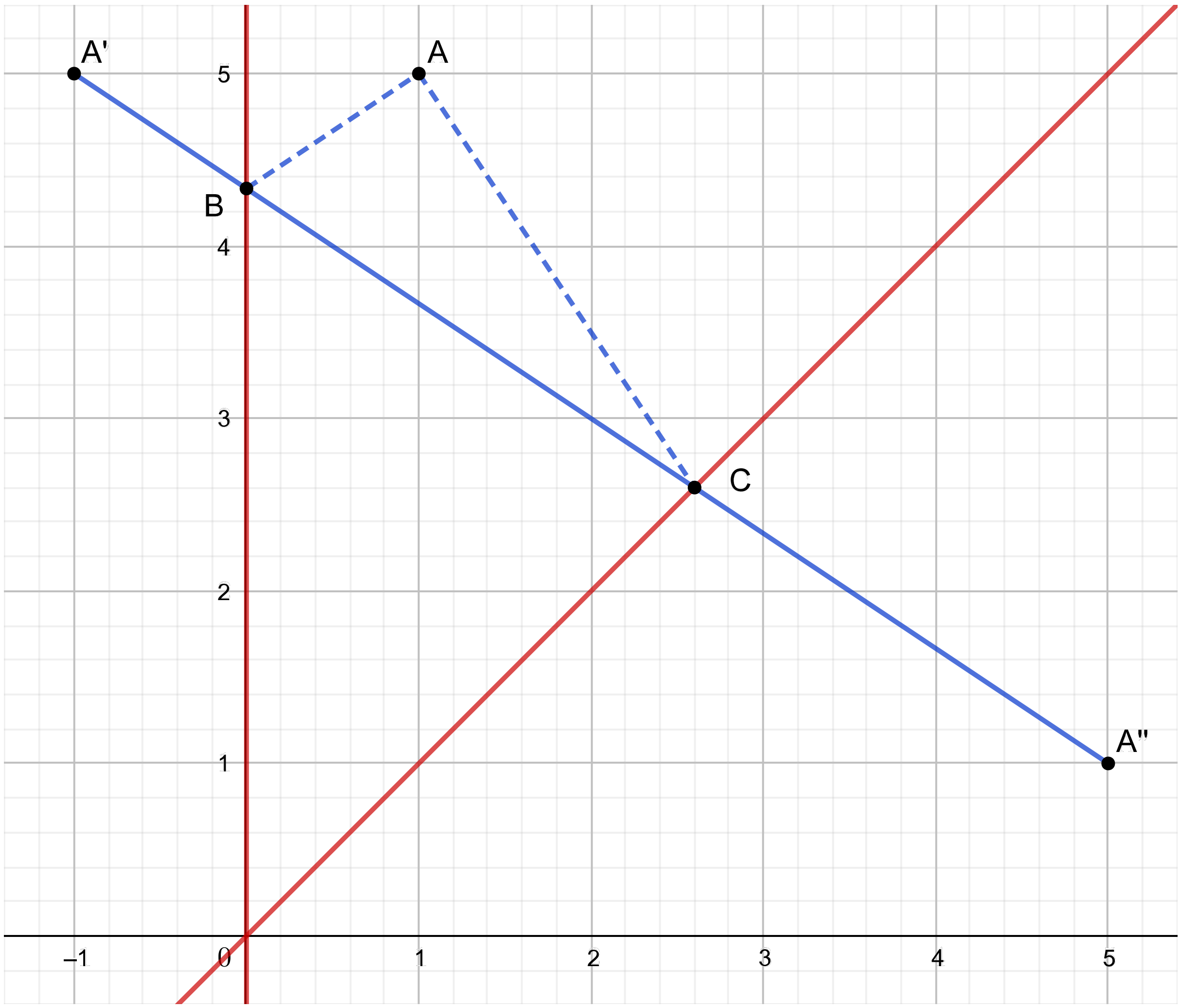
    Solution: Any given person in the group of 5 has 4 options: either they are not in any committee, they are only in committee A, they are in committees A and B or they are in committees A and C. Therefore, without considering that committee A needs at least one person, there are $4^5 = 1024$ possible arrangements.

    Since all committees need at least one person, we need at least one person to pick option 3 and one person to pick option 4. Using PIE counting, the amount of arrangements in which every person is not in option 3 is 3^5. It is the same for the number of arrangements in which every person does not choose option 4. The number of arrangements which exclude both options 3 and option 4 is 2^5, as each of the 5 members has two options. Therefore, the desired total is $4^5 - 3^5 - 3^5 + 2^5 = 570$.

14. On the Cartesian plane, point A is at $(1, 5)$, point B is on the y-axis and point C is on the line $y = x$. What is the minimum possible perimeter of triangle ABC?

Answer: $\boxed{2\sqrt{13}}$

Solution: Consider the diagram shown, in which point A has been reflected across the y-axis to get A', and also across $y = x$ to get A''.



Note that $AB = A'B$ and $CA = CA''$. Therefore, the perimeter of the triangle is equal to $A'B + BC + CA''$.

Thus, the shortest distance between A' and A'' is also the smallest perimeter of triangle ABC.

Applying the distance formula yields $\sqrt{(5-1)^2 + (5-(-1))^2} = \sqrt{52} = 2\sqrt{13}$.

15. Let $\varphi(n)$ denote the number of positive integers at most n which are relatively prime to n. Denote, by S, the sum

$$S = \sum_{i=1}^{2021^{11}} \varphi(\gcd(i, 2021^{11})).$$

Compute the number of positive divisors of S.

Answer: $\boxed{24000}$

Solution: For brevity, let $n = 2021^{11}$. For $d = \gcd(i, n)$, there are $\varphi(n/d)$ multiples of d for which i could exist. Therefore, the sum is equivalent to

$$\sum_{i=1}^{n} \varphi(\gcd(i, n)) = \sum_{d|n} \varphi(n/d)\varphi(d).$$

Let $f(n)$ denote the righthand sum. Note that φ is a multiplicative function, so it is not hard to see that f is multiplicative; more rigorously, if you are familiar with Dirichlet Convolutions, $f(n) = (\varphi * \varphi)(n)$. For $n = p^k$,

$$\begin{aligned} f(p^k) &= 2p^{k-1}(p-1) + p^{k-2}(p-1)^2(k-1) \\ &= p^{k-2}(p-1)(2p + (p-1)(k-1)). \end{aligned}$$

Therefore, for $n = 2021^{11}$,

$$\begin{aligned} f(2021^{11}) &= f(43^{11})f(47^{11}) \\ &= (43^9 \cdot 42 \cdot 506) \cdot (47^9 \cdot 46 \cdot 554) \\ &= 2^4 \cdot 3 \cdot 7 \cdot 11 \cdot 23^2 \cdot 277 \cdot 43^9 \cdot 47^9. \end{aligned}$$

Therefore, S has $5 \cdot 2^4 \cdot 3 \cdot 10^2 = 24000$ factors.