RAMC 2023

High School Individual Round

- **SCORING:** The first 10 questions are worth 1 point each, and the last 5 questions are worth 2 points each, for a total of 20 possible points.
- This round contains 15 questions to be solved in 45 minutes. All answers are integers.
- No computational aids are permitted other than scratch paper, graph paper, and a pen/pencil. No calculators of any kind are allowed.
- Fill out your information, and sign/initial the honor code on the answer sheet provided.
- If you believe there is an error on the test, submit a challenge to the proctors. Please include your name, level (Elem I/II, MS, HS), and explanation of the problem and your solution.

Do not flip the page until the proctor begins the round!

1. Let N be a subset of $\{1, 2, 3, \cdots, 100\}$ such that no two elements of N sum to an integer that is divisible by 9. What is the maximum number of elements of N?

2. Six RMC counselors are standing around a volleyball court. Each counselor is either on the Red Team, Blue Team, or sitting on the bench. Given that each team has at least one member, how many possible ways are there to make teams?

3. There are 16 tennis players on the Century High School team, each with their own unique racket. Suppose the coach were to take each of the players' rackets and randomly return one racket to each player. How many players do we expect to receive their own racket back?

4. When written in base 27, the number $N!$ terminates in 23 zeros. What is the smallest such possible value of N, expressed in base 10?

5. Find the integer value of $\sqrt{20 \cdot 21 \cdot 22 \cdot 23 + 1}$

6. Matthew and Dan flip a fair coin repeatedly heads or tails until one of them wins. Matthew wins once the coin lands two consecutive heads. Dan wins if the coin lands tails followed consecutively by heads. Let p be the probability Matthew wins and q be the probability Dan wins. If $pq = \frac{a}{b}$ for relatively prime positive integers a and b, find $a + b$.

7. Equilateral triangle ABC is inscribed in circle O. Point D is on minor arc AB such that $AD = 6$ and $BD = 10$. Find the length of segment DC.

8. The value of 17! is $3556874280AB000$, where A and B are digits. Find the two-digit number AB.

9. The total sum of the volume, surface area, and edges of a box with integer side lengths is 332. If the longest distance between any two points in the box can be expressed as $\sqrt{n}$, find n.

10. Let $n = 2^{20} \cdot 3^{23}$. How many factors of n^2 are less than n but not a factor of n?

11. The real solutions to the equation $4x^4 + 4 = 31x^3 + 31x$ are of the form $a \pm \sqrt{b}$. Find $a + b$.

12. If $(\cos 1°)(\cos 3°)(\cos 5°) \cdots (\cos 87°)(\cos 89°) = 2^n$, compute $2n$.

13. Let F_n denote the n-th Fibonacci number $(F_1 = 1, F_2 = 1, F_3 = 2, \cdots)$, and let

$$S = \sum_{n=1}^{\infty} \frac{F_n}{2^n} = \frac{F_1}{2^1} + \frac{F_2}{2^2} + \frac{F_3}{2^3} + \cdots .$$

Find $1000S$.

14. Suppose that

$$S = \sum_{n=1}^{1012} (2n-1)^2 \cdot (2025 - 2n) \cdot \binom{2024}{2n-1}.$$

What is the largest value of k such that $\frac{S}{2^k}$ is an integer?

15. For positive real numbers a, b, c,

$$11 = \sqrt{a^2 - ab + b^2},$$
$$22 = \sqrt{a^2 - ac + c^2},$$
$$33 = \sqrt{b^2 + bc + c^2}.$$

If $bc = \frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$.