2nd Rochester Area Math Competition 2020

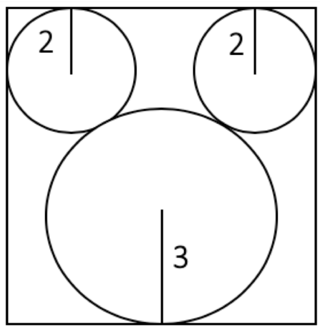
12 September 2020

High School Team

1. An artificial intelligence dataset consists of 200 photos of airplanes, 300 photos of water skis, 250 photos of boats, 300 photos of trains, and 400 photos of spacecraft. Suppose that a batch of n images is selected from the dataset at random. What is the minimum n that will guarantee that at least one photo of each type of object is present?

2. Two lines l_1 and l_2 are perpendicular and intersect at point $(2, 4)$ in the xy coordinate plane. A third line l_3 intersects l_1 at (a, a) and intersects l_2 at $(a + 10, 2a + 9)$. Suppose that $a < 0$. Compute the area of a triangle whose vertices are the points that lie on exactly two lines out of l_1, l_2, and l_3.

3. Three circles are placed inside a square as shown. Two of the circles have radius 2 and are tangent to two sides of the square. These two circles are also externally tangent to the circle with radius 3, which is tangent to the bottom side of the square. Let x be the side length of the square. Then $5x$ can be expressed in the form $-m + n\sqrt{q}$, where m,n, and q are positive integers and q has no divisors that are perfect squares other than 1. Compute $m + n + q$.



4. A virus breaks out on the faraway exoplanet of *Dystopia* in 2050. Suppose that 1 out of every 20 people in the population currently has the virus, which is asymptomatic. Currently, an antibody test is the only available method for detecting the virus. The antibody test will return positive 99% of the time for a person who has the virus. For a person who does not have the virus, the antibody test will accurately return negative 95% of the time and inaccurately return positive 5% of the time. A person from the population is randomly selected, tested, and their result is positive. The probability that they actually have the virus can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

5. The first three terms of an arithmetic sequence are $8 - 8k$, $3k^2$, and $k^4 + 2k^3$ in that order. Additionally, k and 1 can be the first and second terms, respectively, of a convergent infinite geometric series. What is the sum of all possible values of k?

6. Dylan purchases a cube of ice cream from the store that measures 10 inches on each side. He coats the ice cream cube with a thick exterior layer of chocolate that occupies all points that are located within 3 inches of the cube's surface. The volume of chocolate, in cubic inches, can be represented in the form $a + b\pi$, where a and b are integers. Compute $a + b$.

7. Holden synthesizes N identical coins from a rock of meteorite. Each coin lands heads with 75% probability and tails with 25% probability. He then flips all N coins. If they all land heads, he will stop. Otherwise, he will flip all of them again, repeating this procedure until all of them land heads at the same time, at which point he stops. Let $E(N)$ be the expected number of times that he will have to flip each coin. What is the smallest value of N such that $E(N) \geq 4$?

8. Dr. Hu is standing at (0,0) in the coordinate plane and wants to travel to (6,6). She can move to the right or move up in one unit increments. However, there are three teleportation machines located at (1,5), (2,4), and (4,2). If she lands on a teleportation machine, she can choose to move normally (one unit to the right or upward) or she can teleport to another teleportation machine. Once she uses teleportation, all teleportation machines vanish and can no longer be used. Find the number of ways that she can travel from the origin to $(6, 6)$.

9. Functions f and g are defined as $f(x) = x^2 + 2x + 2$ and $g(x) = x^2 - 2x$. Define the polynomial $P(x)$ as follows:

$$P(x) = f(g(f(g(f(g(f(x))))))) - g(f(g(f(g(f(g(x))))))).$$

Let n be the degree of $P(x)$ and let $r_1, r_2, ..., r_n$ be the complex and not necessarily distinct roots of the polynomial $P(x)$. Compute the following:

$$\left| \sum_{1 \leq i < j \leq n} r_i r_j \right|.$$

10. A wizard lands on a random point inside a square-shaped arena. The sides of the arena are laced with poisonous chemicals, and in order to survive, the wizard must land closer to the center of the square, which houses a fountain of regeneration, than the sides of the arena. The probability that the wizard survives can be expressed in the form $\frac{a\sqrt{b}-c}{3}$, where a,b, and c are positive integers and b is not divisible by the square of any prime. Compute $a + b + c$.