RAMC 2022

High School Tiebreaker Round

- **SCORING:** The questions in this round are used to break ties, and do not count towards overall scores.
- This round contains 10 questions. Problems towards the end tend to be more difficult than problems toward the beginning.
- No computational aids are permitted other than scratch paper, graph paper, and a pen/pencil. No calculators of any kind are allowed.
- All answers must be in a reasonably simplified form.
- Fill out your information, and sign/initial the honor code on the answer sheet provided.
- If you believe there is an error on the test, submit a challenge to the proctors. Please include your name, level (Elem I/II, MS, HS), and explanation of the problem and your solution.

Do not flip the page until the proctor begins the round!

1. What is the remainder when $(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)(2^{32}+1)(2^{64}+1)(2^{128}+1)$ is divided by 13?

2. Triangle MAN has a right angle at A. The altitude from A intersects MN at point B. Given that the side lengths of triangle MAN are all positive integers and $BN = 17^3$, find $\tan(\angle MNA)$.

3. Find the smallest positive integer n such that 2022^{2022} is a divisor of $n!$.

4. How many nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ exist such that no two elements in the subset share a common factor greater than 1?

5. A sequence $\{a_n\}$ is defined by $a_0 = \sqrt{3}$ and for $n \geq 1$,

$$a_n = \frac{1}{1 - a_{n-1}} - \frac{1}{1 + a_{n-1}}$$

   Compute a_{2022}.

6. 2022 ants are placed on a 100-meter-long stick, each of them either facing left or right. Each ant moves at a speed of 4 meters per second in the direction they were initially facing. When two ants collide, they both reverse their directions and walk at the same rate. What is the shortest time one must wait to guarantee all ants have walked off the stick, regardless of the starting positions of the ants?

7. How many bijective functions $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ exist such that $f(f(f(x))) \neq x$ for all $1 \leq x \leq 6$?

8. Triangle ABC has perimeter 36 and side lengths a, b, c in arithmetic progression. The value of $a^3 + b^3 + c^3 - 3abc$ can be bounded strictly above by integer n. Find the smallest possible value of n.

9. Let P denote the product

$$P = (4^2 + 64)(9^2 + 64)(16^2 + 64) \cdots (81^2 + 64)(100^2 + 64).$$

   Compute the smallest integer n such that P/n is a perfect square.

10. Initially $n = 1$. Each second, n is multiplied by a factor $k \in \{1, 8, 27\}$, chosen uniformly at random. The probability that $n - 1$ is a multiple of 2021 after 30 seconds can be expressed as $\frac{a}{3^b}$. Find ab.