RAMC 2022

High School Tiebreaker Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. What is the remainder when $(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)(2^{32}+1)(2^{64}+1)(2^{128}+1)$ is divided by 13?

   Answer: $\boxed{2}$

   Solution: When multiplying this expression by $(2-1)=1$, the difference of squares multiplies out so that this value equals $2^{256}-1$. The remainder when 2^n is divided by 13 must repeat in cycles of 12 due to Fermat's Little Theorem, and $256 = 4 \pmod{12}$. We calculate that $2^4 = 3 \pmod{13}$. Thus $2^{256} - 1 = 2 \pmod{13}$.

2. Triangle MAN has a right angle at A. The altitude from A intersects MN at point B. Given that the side lengths of triangle MAN are all positive integers and $BN = 17^3$, find $\tan(\angle MNA)$.

   Answer: $\boxed{\frac{144}{17}}$

   Solution: To find the value of $\tan(\angle MNA)$ using triangle BNA, we must solve for the value $\tan(\angle MNA) = \frac{AB}{17^3}$. By applying the Pythagorean Theorem on triangle BNA, we have $AN^2 - AB^2 = 17^6 = (AN + AB)(AN - AB)$. Because it is given that the triangle lengths are all positive integers, by process of elimination $AN + AB = 17^4$ and $AN - AB = 17^2$. Then, we get $2AB = 17^4 - 17^2$, and $\tan(\angle MNA) = \frac{17^4-17^2}{2\cdot 17^3} = \frac{17^2(17^2-1)}{2\cdot 17^3} = \frac{144}{17}$.

3. Find the smallest positive integer n such that 2022^{2022} is a divisor of $n!$.

   Answer: $\boxed{679729}$

   Solution: Since $2022^{2022} = 2^{2022} \cdot 3^{2022} \cdot 337^{2022}$, we essentially need to find the smallest $n!$ such that there are 2022 factors of 337 in $n!$. Note that $(337 \cdot 2022)!$ has $2022 + \frac{2022}{337} = 2028$ factors of 337. Counting down by multiples of 337, we see that $(337 \cdot 2021)!$ has $2028 - 2 = 2026$ factors of 337 (since 2022 is a multiple of 337), $(337 \cdot 2020)!$ has $2026 - 1 = 2025$ factors, and so on. Thus $(337 \cdot 2017)!$ has 2022 factors of 337, and everything less than $(337 \cdot 2017)!$ has less than 2022. Therefore, $n = 337 \cdot 2017 = 679729$.

4. How many nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ exist such that no two elements in the subset share a common factor greater than 1?

   Answer: $\boxed{71}$

   Solution: Note that 6 is the only number in the set with more than one distinct prime factor, so we do casework based on whether 6 is in the subset or not.

   If 6 is not in the subset, there are four options for having a multiple of 2 in the subset (2, 4, 8, or nothing), two options for a multiple of 3 (3 or nothing), two options for 5, two options for 7, and two options for 1. This makes $4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$, but we overcount the empty set, so there are 63 subsets in this case.

   If 6 is in the subset, we can't have any other multiples of 2 or 3, but there are still two options for 5, two options for 7, and two options for 1. This makes $2 \cdot 2 \cdot 2 = 8$ in this case.

   In total, there are $63 + 8 = 71$ subsets.

5. A sequence $\{a_n\}$ is defined by $a_0 = \sqrt{3}$ and for $n \geq 1$,

$$a_n = \frac{1}{1 - a_{n-1}} - \frac{1}{1 + a_{n-1}}$$

   Compute a_{2022}.

   Answer: $\boxed{\sqrt{3}}$

   Solution: Note that the right hand side is equivalent to $\frac{2a_{n-1}}{(1-a_{n-1}^2)}$. Therefore, if $a_{n-1} = \tan(x)$, then $a_n = \tan(2x)$. In particular, since $a_0 = \tan(\frac{\pi}{3})$, then $a_n = \tan(2^n \cdot \frac{\pi}{3})$. Therefore, $a_{2022} = \tan(2^{2022} \cdot \frac{\pi}{3}) = \tan(\frac{4\pi}{3}) = \sqrt{3}$.

6. 2022 ants are placed on a 100-meter-long stick, each of them either facing left or right. Each ant moves at a speed of 4 meters per second in the direction they were initially facing. When two ants collide, they both reverse their directions and walk at the same rate. What is the shortest time one must wait to guarantee all ants have walked off the stick, regardless of the starting positions of the ants?

   Answer: $\boxed{25}$

   Solution: Note that the collision process is equivalent to if the ants simply passed through one another. Therefore, we can ignore the collision process, and the longest time before an ant falls off is $100 \div 4 = 25$.

7. How many bijective functions $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ exist such that $f(f(f(x))) \neq x$ for all $1 \leq x \leq 6$?

   Answer: $\boxed{720}$

   Solution: We can represent the function as a disjoint union of directed cycles. The given condition $f(f(f(x))) \neq x$ is equivalent to saying there are no 1-cycles or 3-cycles. In particular, this means we can either have:

   Three 2-cycles: $\frac{2^3}{3!} \cdot \binom{6}{2}\binom{4}{2}\binom{2}{2} = 120$,

   A 2-cycle and 4-cycle: $2^2 \cdot \binom{6}{2} \cdot (4-1)! = 360$,

   Or a 6-cycle: $2 \cdot (6-1)! = 240$.

   Therefore, there are $120 + 360 + 240 = 720$ such f.

8. Triangle ABC has perimeter 36 and side lengths a, b, c in arithmetic progression. The value of $a^3 + b^3 + c^3 - 3abc$ can be bounded strictly above by integer n. Find the smallest possible value of n.

   Answer: $\boxed{3888}$

   Solution: From Newton's sums, there exists the identity

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2).$$

   Letting $a = b - d$ and $c = b + d$, where d is the common difference, then since $a + b + c = 36$, $b = 12$, so

$$\frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2) = 108d^2.$$

   For a non-degenerate triangle, we need $a + b > c$, which is satisfied when $d < 6$. Thus, the value is bounded strictly above by $108 \cdot 6^2 = 3888$.

9. Let P denote the product

$$P = (4^2 + 64)(9^2 + 64)(16^2 + 64) \cdots (81^2 + 64)(100^2 + 64).$$

Compute the smallest integer n such that P/n is a perfect square.

Answer: $\boxed{3145}$

Solution: From Sophie Germain identity, the product is equivalent to

$$\prod_{k=2}^{10} \left(k^4 + 4 \cdot 2^4\right) = \left(k^2 - 4k + 8\right)\left(k^2 + 4k + 8\right) = \left((k-2)^2 + 4\right)\left((k+2)^2 + 4\right).$$

Note that for factors $(4^2 + 4) \cdots (8^2 + 4)$, they are multiplied twice in the product, so it suffices to consider $(0^2 + 4) \cdots (3^2 + 4)(9^2 + 4) \cdots (12^2 + 4)$.

Expanding the product, it is clear that the minimum such value is $5 \cdot 17 \cdot 37 = 3145$.

10. Initially $n = 1$. Each second, n is multiplied by a factor $k \in \{1, 8, 27\}$, chosen uniformly at random. The probability that $n - 1$ is a multiple of 2021 after 30 seconds can be expressed as $\frac{a}{3^b}$. Find ab.

Answer: $\boxed{930}$

Solution: Note that modulo 2021, $27 = 2^{11}$, $8 = 2^3$, and $1 = 2^0$. Letting x, y, z denote the number of times $27, 8, 1$ are multiplied, respectively, it then suffices to find the number of sequences such that $2^{11x+3y+0z} = 1 \pmod{2021}$.

$2021 = 43 \cdot 47$, so the order of $2 \pmod{43}$ must divide 42, for which it is not hard to compute that $\text{ord}_{43}(2) = 14$. Likewise, the order of $2 \pmod{47}$ must divide 46, so it is not hard to check the divisors of 46 and verify $\text{ord}_{47}(2) = 23$. Therefore, $23 \cdot 14 = 322$ must divide $11x + 3y$, and $x + y + z = 30$, or equivalently, $x + y \leq 30$.

Case 1: $11x + 3y = 0$. This is trivially the case where n is multiplied by 1 thirty times, which has only one way.

Case 2: $11x + 3y = 322$. Note that $(x, y) = (29, 1)$ is a solution and that (x, y) satisfies $11x + 3y = 22$ if and only if $(x - 3, y + 11)$ does. Since y must be nonnegative, and $x - 3 + y + 11 > x + y$, then $(29, 1)$ has the smallest sum $x + y$. Therefore, $(x, y, z) = (29, 1, 0)$ is the only solution, which has 30 possible sequences.

Thus, the probability is $\frac{31}{3^{30}}$, meaning $ab = 31 \cdot 30 = 930$.