RAMC 2022

Middle School Individual Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. The Rochester Math Club employs 2022 employees. If its employees form n equal groups, what is the total number of choices for n?

   Answer: $\boxed{8}$

   Solution: If a group has k people, then $n = \frac{2022}{k}$. Therefore, this problem is essentially asking for the number of factors of 2022. $2022 = 2 \cdot 1011 = 2 \cdot 3 \cdot 337$, so there are 8 factors (1, 2, 3, $2 \cdot 3$, 337, $2 \cdot 337$, $3 \cdot 337$, and 2022). There are 8 choices for n.

2. Four ounces of gold can be traded for $23\frac{3}{10}$ pounds of silver. Jake wants to know the monetary value of his $\frac{1}{2}$ pound of gold, when the price of silver is \$20 an ounce. If a pound is 16 ounces, how many dollars is Jake's $\frac{1}{2}$ pound of gold worth?

   Answer: $\boxed{14912}$

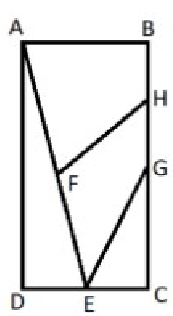
   Solution: We can write out the conversion.

$$\begin{aligned}\frac{1}{2}\text{ lb gold} &= 8\text{ oz gold}\\ &= 46.6\text{ lb silver}\\ &= 745.6\text{ oz silver}\\ &= \$14912.\end{aligned}$$

3. There exists rectangle $ABCD$ such that $\overline{BC} = 2\overline{AB}$. Let E be the midpoint of DC, F be the midpoint of AE, G is the midpoint of BC, and H is the midpoint of BG. Let $[XYZ]$ denote the area of a polygon XYZ. If $[EFHG] = 9$, find $\overline{CH}$.

   Answer: $\boxed{6}$

   Solution: Let the area of $ABCD = A$. As $FG||DC$, and since F is the midpoint of AE, $\overline{FG} = \frac{3}{4}\overline{CD}$. The height is $\frac{1}{2}$ of the rectangles height, so $[\triangle FGE] = \frac{3}{16}A$. $\triangle FGH$ can be calculated similarly, and since $\overline{HG} = \frac{1}{4}\overline{BC}$, $[\triangle FGH] = \frac{3}{32}A$. This gives us $(\frac{3}{16} + \frac{3}{32})A = 9$, or $A = 32$. This means $\overline{AB} = 4$ and $\overline{BC} = 8$, and $\overline{CH} = \frac{3}{4}\overline{BC}$, so $\overline{CH} = 6$.



4. Five pizzas were ordered for a party. Each pizza is cut into x slices. The party contains 13 people. Each person eats a different, positive number of slices. If each person chooses one of the 5 pizzas to eat from exclusively, what is the smallest possible value of x, assuming that x is an integer?

Answer: $\boxed{19}$

Solution: For 13 people, the minimum amount of total slices must be $1+2+\cdots+13 = \frac{13\cdot 14}{2} = 91$. Therefore, we need at least 91 total slices. However, as this needs to be an integer amount of slices per pizza, the minimum amount of slices must be divisible by 5, making 95 slices, or 19 slices per pizza.

We can satisfy the condition that each person only eats from 1 pizza as we can split the numbers $1, 2, \cdots, 13$ such that all 5 pizzas have ≤ 19. Therefore, the answer is 19.

5. What is the tens digit of 6^{2022}?

Answer: $\boxed{3}$

Solution: We notice a pattern: as the units digit doesn't change, the number of tens digits in a 'cycle' has to be ten or less.

x	1	2	3	4	5	6	7	8	$\cdots$
$6^x \pmod{100}$	6	36	16	96	76	56	36	16	$\cdots$

We have a cycle of 5, as shown above between 6^2 and 6^6. Therefore, $6^{2022} \equiv 6^2 \pmod{100}$, so its tens digit is 3.

6. Carl picks a random integer from 1 to 3, inclusive. He then picks a random number from 1 to 4, 1 to 5, and so on until 1 to 10. Carl then adds together each number that he picked. Out of all possible combinations, what is the average value of this sum?

Answer: $\boxed{30}$

Solution: The average value of a random number from 1 to n is equal to the average of all numbers included. In other words, the expected value for each 'turn' is

$$\frac{1+2+3+\cdots+n-1+n}{n} = \frac{(n)(n+1)}{2n} = \frac{n+1}{2}.$$

Now, we can find the expected value of the whole sum

$$\frac{3+1}{2} + \frac{4+1}{2} + \cdots + \frac{10+1}{2} = \frac{66-6}{2} = 30.$$

Therefore, the expected value is 30.

7. A new clothing store has opened in Rochester and to celebrate their grand opening, they are offering a discount on jeans and sunglasses. On opening day, 100 people buy jeans and 42 people buy sunglasses. Some people buy both. There are 6 people who buy neither. If one of the people that buys jeans is selected at random, the probability that they also buy glasses is $\frac{3}{10}$. How many people attend the store's opening?

Answer: $\boxed{118}$

Solution: Of the 100 people who buy jeans, $\frac{3}{10}$ of them also buy sunglasses, so $\frac{3}{10} \cdot 100 = 30$ people buy both jeans and sunglasses. This means that $42 - 30 = 12$ people buy only sunglasses and $100 - 40 = 60$ people buy only jeans. Therefore, the number of people who attend the grand opening is $40 + 60 + 12 + 6 = 118$.

8. A cylinder and a sphere have the same volume. The cylinder has a diameter of 6 feet and a height of 4 feet. What is the diameter of the sphere, in feet?

Answer: $\boxed{6}$

Solution: As cylinder has a radius of $\frac{6}{2} = 3$ feet and a height of 4 feet. The volume of the cylinder is $3^2 \cdot 4 \cdot \pi = 36\pi$. The volume of the sphere is be $\frac{4}{3}\pi r^3$ where r is the radius of the sphere. Therefore, in order for the two shapes to have the same volume,

$$\begin{aligned} 36\pi &= \frac{4}{3}\pi r^3 \\ 27\pi &= r^3 \\ r &= 3. \end{aligned}$$

Therefore, the diameter of the sphere is $3 \cdot 2 = 6$.

9. What is the value of $(1 + 2 - 3) + (4 + 5 - 6) + \cdots + (2020 + 2021 - 2022)$?

Answer: $\boxed{680403}$

Solution: We can split the expression into two parts.

$$\begin{aligned} (1 + 2 - 3) + \cdots + (2020 + 2021 - 2022) &= 1 + 4 + \cdots + 2020 + (2 - 3) + (5 - 6) + \cdots + (2021 - 2022) \\ &= \frac{2021 \cdot 674}{2} + (-1) \cdot 674 \\ &= 2021 \cdot 337 + (-2) \cdot 337 \\ &= 2019 \cdot 337 \\ &= 680403. \end{aligned}$$

10. Find the coefficient of x^{18} in the expanded expression for $(x^3 - x^2 + 4x)^5(6x^2 + 5x + 13)^2$.

Answer: $\boxed{-120}$

Solution: There are a total of 7 terms before we completely expand it. We can make x^{18} with the following combinations. The total coefficient is calculated by the # of ways for that combination, times the coefficients in the inital expression.

term 1	term 2	term 3	term 4	term 5	term 6	term 7	total coefficient
3	3	3	3	3	2	1	$2 \cdot 1^5 \cdot 6 \cdot 5 = 60$
3	3	3	3	2	2	2	$5 \cdot 1^4 \cdot (-1) \cdot 6^2 = -180$

Therefore, the coefficient is $60 - 180 = -120$.

11. Bob is selling books. He has 20 hardcover books and 16 paperback books. A hardcover book costs \$21, and a paperback book costs \$17. If Bob earned a total of \$360, how many paperback books did he sell?

Answer: $\boxed{15}$

Solution: We can write this as an equation: $21x + 17y = 360$. Let x be the number of hardcover books he sells, and y be for paperback books, respectively. We can write y in terms of x.

$$y = \frac{360 - 21x}{17}.$$

To find a value that makes it an integer, we have

$$\begin{aligned} 360 - 21x &\equiv 0 \pmod{17} \\ 20 - 4x &\equiv 0 \pmod{17} \\ x &= 5. \end{aligned}$$

As $x = 5$, we have $y = \frac{360-105}{17} = 15$. There are no other solutions that work, as we do have a limit of the amount of both books. Therefore, Bob sold 15 paperback books.

12. In isosceles triangle $\triangle ABC$, $\overline{AB} = \overline{BC}$. D is located on line BC such that $\overline{BD} : \overline{DC} = 3 : 2$. M is the midpoint of line AC. Point E is the intersection of lines AD and BM. Let $[XYZ]$ denote the area of a polygon XYZ. If the $[\triangle BAM] = 10$, what is $[\triangle ADC]$?

    Answer: $\boxed{8}$

    Solution: Triangles $\triangle BAM$ and $\triangle BMC$ have the same height from B, and $\overline{AM} = \overline{MC}$, so they have the same area. The total area of $\triangle ABC$ then equals

$$[\triangle BAM] + [\triangle BMC] = 10 + 10 = 20.$$

In triangles $\triangle ABD$ and $\triangle ADC$, the height from A is the same, so $\dfrac{[ABD]}{[ADC]} = \dfrac{\overline{BD}}{\overline{DC}} = \dfrac{3}{2}$. Additionally, $[\triangle ABD] + [\triangle ADC] = 20$. Therefore, if the area of triangle ADC is x, we have

$$\begin{aligned} \frac{20 - x}{x} &= \frac{3}{2} \\ 40x - 2x &= 3x \\ x &= 8. \end{aligned}$$

13. Denise has ten coins. Nine of the coins are fair. The tenth coin is weighted, and has a 75% chance of landing heads. Denise picks one coin at random, and flips it three times. The coin lands heads all three times. What is the probability that the coin she picked was weighted?

    Answer: $\boxed{\frac{3}{11}}$

    Solution: Denise had a $\frac{1}{10}$ chance of picking the weighted coin, and the coin has a $(\frac{3}{4})^3 = \frac{27}{64}$ chance of landing three heads in a row. Denise has a $\frac{9}{10}$ chance of picking a fair coin, and they have a probability of $(\frac{1}{2})^3 = \frac{1}{8}$ of landing three heads in a row.

    By Bayes' theorem, we can calculate

$$\begin{aligned} P(\text{coin Denise picked was weighted}) &= \frac{\frac{1}{10} \cdot \frac{27}{64}}{\frac{1}{10} \cdot \frac{27}{64} + \frac{9}{10} \cdot \frac{1}{8}} \\ &= \frac{27}{27 + 72} \\ &= \frac{3}{11}. \end{aligned}$$

14. The six-digit number $\underline{5}\ \underline{5}\ \underline{A}\ \underline{4}\ \underline{7}\ \underline{B}$ is divisible by 44. What is the remainder when the number is divided by 9?

Answer: $\boxed{3}$

Solution: As the number needs to be divisible by 4 and 11, we'll tackle the 4 part with the cases, and the 11 part within the cases. We note that the only two numbers $7B$ that are divisible by 4 are 72 and 76.

Case 1: $B = 2$. $55A472$ needs to be divisible by 11. This means that

$$55A472 \equiv 5 + A + 7 - 5 - 4 - 2 \equiv A + 1 \equiv 0 \pmod{11}.$$

There is no digit-value that is possible for A.

Case 2: $B = 6$. Similarly,

$$55A476 \equiv 5 + A + 7 - 5 - 4 - 6 \equiv A - 3 \equiv 0 \pmod{11}.$$

This means that $A = 3$. Therefore, the number is 553476, and its remainder when divided by 9 is $553472 \equiv 5 + 5 + 3 + 4 + 7 + 6 \equiv 30 \equiv 3 \pmod 9$.

15. Triangle $\triangle ABC$ is circumscribed by a circle, with $\overline{AB} = 6$, $\overline{AC} = 8$, and $\angle A = 90°$. Extend the angle bisector from A until it intersects the circle again at point D. What is the length of $\overline{CD}$?

Answer: $\boxed{5\sqrt{2}}$

Solution: As $\angle BAC = 90°$, $\angle BAD = \angle DAC = 45°$. We also know that BC is the diameter, as $\angle BAC = 90°$. This means arc $\overset{\frown}{BD}$ and $\overset{\frown}{CD}$ are equal in length. This also means that OD is perpendicular to BC, as the central angle $\angle DOC = 2 \cdot 45 = 90°$. As the radius is $\sqrt{6^2 + 8^2} = 10$, this means that $DC = \sqrt{DO^2 + OC^2} = \sqrt{50} = 5\sqrt{2}$.