Rochester Area Math Competition Solutions (Middle School)

Hosted by Rochester Math Club (RMC)

March 14th, 2018

Not to be posted until **March 14th, 2018**, after 8:30pm CST.

1 Solutions

1. $4(7^2 + x) = 28 - 12x$. Find x.

Solution: $-\mathbf{\dfrac{21}{2}}$
Divide both sides of the equation by 4.

$$\begin{aligned}7^2 + x &= 7 - 3x \\ 49 - 7 &= -3x - x \\ 42 &= -4x \\ x &= \frac{-21}{2}\end{aligned}$$

2. In the arithmetic sequence 107, 88, 69, ..., what is the 7th term?

Solution: -7
The common difference d is $88 - 107 = -19$. Using the explicit formula for arithmetic sequences, $a_7 = 107 + (7 - 1)(-19) = -\mathbf{7}$.

3. If the degree measures of the angles of a triangle are in the ratio $2 : 3 : 5$, what is the degree measure of the largest angle of the triangle?

Solution: 90°
$2x + 3x + 5x = 180$, $x = 18$. The largest angle is $5 \times 18 = \mathbf{90}°$.

4. Tracy goes to a friend's house. Her friend has some cats and some chickens. Tracy counts that there are 11 animals in total, and they have 28 legs in total. Each cat has 4 legs, and each chicken has 2 legs. What is the positive difference between the number of cats and the number of chickens Tracy's friend has?

Solution: 5
Let the number of cats be x and number of chickens be y. Set up the equations:

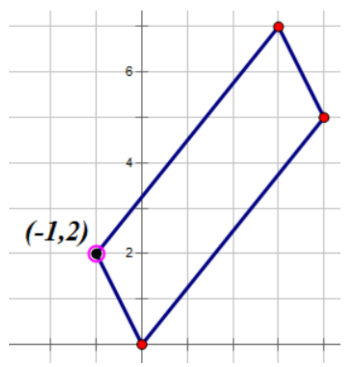
$$\begin{aligned}x + y &= 11 \\ 4x + 2y &= 28\end{aligned}$$

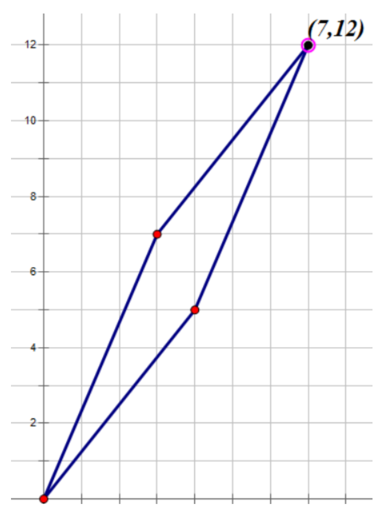
Multiply the first equation by 2 to get $2x + 2y = 22$. Subtract this equation from the second one to get $2x = 6$, $x = 3$. There are 3 cats. $3 + y = 11$, $y = 8$. There are 5 chickens. The positive difference between them is **5**.

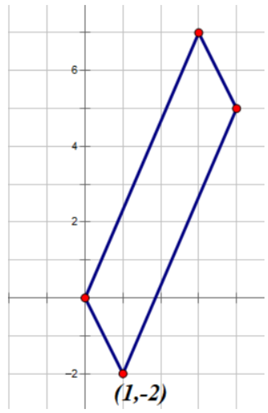
5. A triangle has side lengths 9, 40, and 41. Find the shortest altitude in this triangle.

Solution: $\mathbf{\frac{360}{41}}$
This triangle is a right triangle ($9^2 + 40^2 = 41^2$). The shortest altitude is the altitude to the longest side, which is the hypotenuse with a length of 41. Call this altitude a, and use the area formula to set up $\frac{1}{2}(9)(40) = \frac{1}{2}(41)(a)$. Solve the equation to get $a = \mathbf{\frac{360}{41}}$.

6. A parallelogram in the coordinate plane has coordinates $(0, 0)$, $(3, 7)$, $(4, 5)$, and (x, y). Find the sum of all possible values of x.
Solution: 7
After plotting the three given points, connect them to form sides of a parallelogram. There are three ways to connect them, so there are three possibilities for the fourth point. The fourth point is found by considering the slopes of the sides.







Adding the three x coordinates, $-1+7+1=\mathbf{7}$.

7. If there are 5 identical balls and 3 identical boxes, how many ways can you put the balls in the boxes if you can leave boxes empty, but all balls have to be stored?

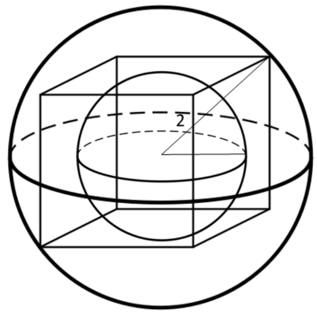
Solution: 5
Divide into 3 cases: one box has all the balls, two boxes have all the balls, or all three boxes contain one or more balls.
First case: there is only 1 possibility for one box to have all the balls.
Second case: you can divide the 5 balls into $1,4$ or $2,3$ for the two boxes, so 2 additional possibilities.
Third case: you can divide the 5 balls into $1,1,3$ or $1,2,2$ for the three boxes, so 2 additional possibilities.
In total, there are $1+2+2=\mathbf{5}$ possibilities.

8. How many distinct positive 3-digit integers satisfy the condition that the product of their digits is 72?

Solution: 24
72 has the single-digit factors of $1,2,3,4,6,8,9$.
If we start with 1, the other two digits must be 8 and 9. 189 has $3!=6$ permutations.
Moving on to 2, the other two digits should have a product of 36. The possibilities are $4,9$ and $6,6$. 249 has 6 permutations, and 266 has $\frac{3!}{2!}=3$ permutations.
If we start with 3, the other two digits have a product of 24. The possibilities are $3,8$ and $4,6$. 338 has 3 permutations, and 346 has 6 permutations.
All the possibilities starting with $4,6,8,9$ have been accounted for by the different permutations. In total, there are $6+6+3+3+6=\mathbf{24}$ possibilities.

9. A cube is inscribed in a sphere of radius 2, and another sphere is inscribed in the cube. What is radius of the smaller sphere?

Solution: $\mathbf{\frac{2\sqrt{3}}{3}}$



The radius of the larger sphere is equivalent to half of the diagonal of the cube. The radius of the smaller sphere is half of the side length of the cube. The diagonal of the cube is $\sqrt{3s^2} = 4$, $3s^2 = 16$, $s^2 = \frac{16}{3}$, $s = \frac{4\sqrt{3}}{3}$. The radius of the smaller sphere is $\mathbf{\frac{2\sqrt{3}}{3}}$.

10. If 363 in base b is equal to 243 in base 10, what is b?

Solution: 8

$$\begin{aligned} 3b^2 + 6b + 3 &= 243 \\ 3b^2 + 6b - 240 &= 0 \\ b^2 + 2b - 80 &= 0 \\ (b-8)(b+10) &= 0 \\ b &= 8, -10 \end{aligned}$$

-10 is extraneous, so the answer is **8**.

11. Find the remainder when the last two digits of 11^9 are divided by 6.

Solution: 1

Solution A: We use mods;

$$11 \equiv 5 \mod 6$$

For 5^n where $n \geq 2$, the last two digits will always be 25, so the remainder when 6 divides 25 is simply **1**.

Solution B: Let's find a pattern. Let's continue to raise the power of 11, but we can discard everything but the last two digits.

$$11^1 = 11$$

$$11^2 = 121 = 21$$
$$11^3 = 1331 = 31$$
$$11^4 = 14641 = 41$$
$$\vdots$$
$$11^9 = \ldots 91$$

So we just need to find the remainder when 6 divides 91, which is simply 1.

12. Richard and Tracy are playing a dice game. If a player rolls $1-5$, it changes to the other player's turn. If a player rolls a 6, s/he rolls again. The first person to roll two 6's in a row in his/her turn wins. If Richard goes first in this game, what is the probability of him winning with no prior 6's rolled by him or Tracy?

Solution: $\mathbf{\frac{1}{11}}$ The probability of Richard rolling two 6's in his first turn is $(\frac{1}{6})^2$.
If he wins in his second turn with no prior 6's rolled, then every roll before must be $1-5$. The probability is $(\frac{5}{6})^2(\frac{1}{6})^2 = \frac{5^2}{6^4}$.
The probability of Richard winning in his third turn is $(\frac{5}{6})^4(\frac{1}{6})^2 = \frac{5^4}{6^6}$.
We see that with each extra turn Richard receives, the probability of him winning is the previous probability multiplied by $\frac{5^2}{6^2} = \frac{25}{36}$. This results in an infinite series. The total probability is

$$\frac{\frac{1}{36}}{1-\frac{25}{36}} = \mathbf{\frac{1}{11}}$$

13. Given $a+b=3$ and $a^2+b^2=12$, find exactly the value of a^6+b^6.

Solution: 1647
We square the first equation to find that $a^2+2ab+b^2=9$. Plugging in the second equation, $12+2ab=9$, and $ab=-\dfrac{3}{2}$. We want to find a^6+b^6. Both of these terms would appear if we multiplied out $(a^2+b^2)(a^4+b^4)$.

Expanding,

$$(a^2+b^2)(a^4+b^4) = a^6+a^2b^4+a^4b^2+b^6 = a^6+b^6+(a^2b^2)(a^2+b^2)$$

Thus, the only thing left to find is a^4+b^4, which can simply be found my squaring a^2+b^2.

$$a^2+b^2 = 12$$
$$(a^2+b^2)^2 = 12^2$$

$$a^4 + b^4 + 2a^2b^2 = 144$$

We know $ab = -\frac{3}{2}$, so $a^2b^2 = \frac{9}{4}$. Plugging this in,

$$a^4 + b^4 + \frac{9}{2} = 144$$

and

$$a^4 + b^4 = \frac{279}{2}$$

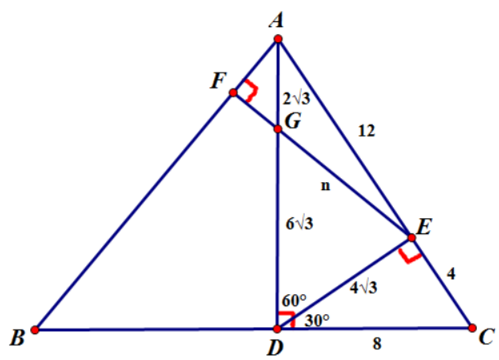
So,

$$(a^2 + b^2)(a^4 + b^4) = 12 * \frac{279}{2} = a^6 + b^6 + ((-\frac{3}{2})^2)(12) = a^6 + b^6 + 9 = 27$$

Doing some multiplication and subtracting both sides by 27, we reach our final answer of **1647**.

14. In triangle ABC, points D, E, F are on BC, CA, AB, respectively, such that $AD \perp BC$, $DE \perp CA$, $EF \perp AB$. AD and EF intersect at G. Given that $CE = 4$, $EA = 12$, and that $AG = 2\sqrt{3}$, find EG.

Solution: $\mathbf{2\sqrt{21}}$



Refer to the figure. We see that since $CE = 4$ and $EA = 12$, $DE = \sqrt{4 \cdot 12} = \sqrt{48} = 4\sqrt{3}$. Now, since $CE = 4$, we notice that ΔDEC is a $30 - 60 - 90$ triangle, and $\angle D = 30°$. Thus, since $AD \perp DC$, then $\angle ADE = 60°$. Further, since $AC = 16$ and $\angle C = 60°$, $AD = 8\sqrt{3}$. We are given that $AG = 2\sqrt{3}$, so $GD = 6\sqrt{3}$.

In the end, we want to find EG, which can now be found using law of cosines on triangle GED.

$$GE^2 = GD^2 + ED^2 - 2(GD)(ED)\cos(GDE)$$

$$GE^2 = (6\sqrt{3})^2 + (4\sqrt{3})^2 - 2(6\sqrt{3})(4\sqrt{3})\cos(60°)$$

$$GE^2 = 108 + 48 - 2(72)(\frac{1}{2}) = 156 - 72 = 84$$

Finally,

$$GE = \sqrt{84} = \mathbf{2\sqrt{21}}$$

15. The roots of the polynomial $x^3 + 4x - 1 = 0$ are r_1, r_2 and r_3. Find exactly

$$\frac{2r_1^2}{(3r_2+1)(3r_3+1)} + \frac{2r_2^2}{(3r_1+1)(3r_3+1)} + \frac{2r_3^2}{(3r_1+1)(3r_2+1)}$$

.

Solution: $\mathbf{\frac{1}{32}}$

Let's just take out the 2 from the expression and multiply it back at the end. Let's also combine denominators.

$$\frac{r_1^2}{(3r_2+1)(3r_3+1)} \cdot \frac{3r_1+1}{3r_1+1} + \frac{r_2^2}{(3r_1+1)(3r_3+1)} \cdot \frac{3r_2+1}{3r_2+1} + \frac{r_3^2}{(3r_1+1)(3r_2+1)} \cdot \frac{3r_3+1}{3r_3+1}$$

So

$$= \frac{3r_1^3 + r_1^2}{(3r_1+1)(3r_2+1)(3r_3+1)} + \frac{3r_2^3 + r_2^2}{(3r_1+1)(3r_2+1)(3r_3+1)} + \frac{3r_3^3 + r_3^2}{(3r_1+1)(3r_2+1)(3r_3+1)}$$

Now we can combine. Grouping,

$$= \frac{3(r_1^3 + r_2^3 + r_3^3) + (r_1^2 + r_2^2 + r_3^2)}{(3r_1+1)(3r_2+1)(3r_3+1)}$$

Looking at this expression, we see that we want to find the sum of the cubes of the roots and also the sum of the squares of the roots. This can be done through Newton Sums and Vieta's, which we get from our equation.

Going back$\Rightarrow$ $x^3 + 4x - 1 = 0$ tells us that (with Vieta's):

$$r_1 + r_2 + r_3 = 0 \quad (1)$$

$$r_1r_2 + r_1r_3 + r_2r_3 = 4 \quad (2)$$

$$r_1r_2r_3 = 1 \quad (3)$$

We first find $r_1^2 + r_2^2 + r_3^2$, which can be done by squaring (1).

$$(r_1 + r_2 + r_3)^2 = r_1^2 + r_2^2 + r_3^2 + 2(r_1r_2 + r_1r_3 + r_2r_3)$$

So

$$0^2 = r_1^2 + r_2^2 + r_3^2 + 2(4)$$

and

$$-8 = r_1^2 + r_2^2 + r_3^2$$

Now we move onto $r_1^3 + r_2^3 + r_3^3$. Instead of cubing the first term, which would be overly-strenuous (and really not fun), we notice that since $r_1 + r_2 + r_3 = 0$, then $r_3 = -r_1 - r_2$.

We can use this fact and substitute it into what we want to find.

$$r_1^3 + r_2^3 + r_3^3 = r_1^3 + r_2^3 + (-r_1 - r_2)^3$$

Expanding the parenthesized value using the binomial theorem (or just hand expansion) finds:

$$r_1^3 + r_2^3 + (-r_1 - r_2)^3 = r_1^3 + r_2^3 - (r_1^3 + 3r_1^2 r_2 + 3r_1 r_2^2 + r_2^3)$$

Multiplying the negative in, some things will cancel; we are left with a simpler expression.

$$r_1^3 + r_2^3 - (r_1^3 + 3r_1^2 r_2 + 3r_1 r_2^2 + r_2^3)$$
$$= r_1^3 + r_2^3 - r_1^3 - 3r_1^2 r_2 - 3r_1 r_2^2 - r_2^3 = -3r_1^2 r_2 - 3r_1 r_2^2$$
$$= -3r_1 r_2 (r_1 + r_2)$$

But again using (1), $r_1 + r_2 = -r_3$, so

$$= -3r_1 r_2 (r_1 + r_2) = -3r_1 r_2 (-r_3) = 3(r_1 r_2 r_3)$$

Using (3), this value is simply 3, so $r_1^3 + r_2^3 + r_3^3 = 3$ We can finally plug these values back in.

$$\frac{3(r_1^3 + r_2^3 + r_3^3) + (r_1^2 + r_2^2 + r_3^2)}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)} = \frac{3(3) + (-8)}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)}$$

$$= \frac{1}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)}$$

The only object left to tackle is the denominator. Simple expansion reveals that $(3r_1 + 1)(3r_2 + 1)(3r_3 + 1) = 27(r_1 r_2 r_3) + 9(r_1 r_2 + r_1 r_3 + r_2 r_3) + 1 = 27(1) + 9(4) + 1 = 64$ So,

$$\frac{1}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)} = \frac{1}{64}$$

But let's not forget about the 2 we factored out at the beginning! Our answer is $2 \cdot \frac{1}{64} = \mathbf{\frac{1}{32}}$.