RAMC 2021

High School Individual Round

- **SCORING:** The first 10 questions are worth 1 point each, and the last 5 questions are worth 2 points each.
- This round contains 15 questions to be solved in 45 minutes. Problems towards the end tend to be more difficult than problems toward the beginning.
- No computational aids are permitted other than scratch paper, graph paper, and a pen/pencil. No calculators of any kind are allowed.
- All answers are integers. When submitting answers, do not add additional characters (such as spaces or units) beyond pure numerical digits, with the exception of a minus (-) symbol when needed.
- If you believe there is an error on the test, submit a challenge to rochestermathclub@gmail.com. Please include your name, level (Elem I/II, MS, HS), and explanation of the problem and your solution.

Take a moment to check that your information is entered correctly!

1. Using a standard 52-card deck of cards, a random set of 5 cards is selected. The probability that this set of cards will contain exactly three cards of the same rank can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. Find $a + b$.

2. For how many values of x does the expression $\sqrt{150 - \sqrt{x}}$ evaluate to an integer?

3. Jacob writes N, a three-digit multiple of 7, on a whiteboard, then covers one of the digits with a piece of paper. Julie knows that N is a multiple of 7. No matter which of the three digits are being covered, if Julie looks at the board, she can be certain of the value of the digit that is being covered. How many possible values of N are there?

4. On triangle ABC, $AB = 4$, $AC = 6$, and $m\angle A = 120°$. The angle bisector of $\angle A$ intersects side AC at L. The length of segment AL can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. What is $a + b$?

5. Suppose $\sin\alpha + \sin\beta = \frac{\sqrt{6}}{3}$ and $\cos\alpha + \cos\beta = \frac{\sqrt{3}}{3}$. The value of $\cos^2\left(\frac{\alpha - \beta}{2}\right)$ can be expressed as a ratio of two relatively prime positive integers, $\frac{m}{n}$. Find $m + n$.

6. Regular pentagon $ABCDE$ has side length 4. Triangle ABD is constructed, and its incircle is drawn, which is tangent to side AD and BD at points P and Q, respectively. The area of triangle PQD can be expressed as $a\sqrt{\frac{1}{2}(b - \sqrt{c})}$, where c is not a multiple of any square number, and a, b, and c are integers. Find $a + b + c$.

7. Roots r_1, r_2, and r_3 of the polynomial $x^3 - 5x^2 - 8x + a$ satisfy the equation $r_1 + 2r_2 + 4r_3 = 0$. What is the sum of the possible values of a?

8. Let $S(n, \theta) = \prod_{i=0}^{n} \sin(2^i\theta)$ and similarly let $C(n, \theta) = \prod_{i=0}^{n} \cos(2^i\theta)$. For any integer N, there exists a pair of integers, (j, k), such that for all values of θ, $S(N - 1, \theta) = 2^j \sin^k(\theta) \prod_{n=0}^{N-2} C(n, \theta)$. When simplified, $j + k = \frac{1}{2}P(N)$, where $P(N)$ is a polynomial of N. Determine the sum of the coefficients of $P(N)$.

9. George labels 1000 quarters with the numbers 1 to 1000. He lays them all on a table, heads facing up. He flips over each coin whose label is a multiple of 3 but not 5. Then, he flips over each coin whose label is a multiple of 5 but not 7. Finally, he flips over each coin whose label is a multiple of 7 but not 11. George now chooses two unique coins on the table. The probability that both of the chosen coins are tails facing up can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. What is $a+b$?

10. How many ordered pairs of integers, (x, y), satisfy the equation $\frac{1}{x^2} - \frac{3}{y} = \frac{1}{25}$?

11. Find the sum of the values of x for which $\log_2 x \log_3 x \log_4 x = \log_2 x(\log_2 x + \log_3 x + \log_4 x)$.

12. How many strictly increasing three-term geometric sequences with integer ratios can be formed by using only the positive integers between 1 and 169, inclusive?

13. For any positive real number x, let $[x]$ denote the greatest integer no larger than x, and let $\{x\}$ denote the decimal portion of x, such that $[x] + \{x\} = x$. Find the smallest possible positive integer n such that $1 - \left[\sqrt{n^2+1} + n\right] \cdot \left\{\sqrt{n^2+1} + n\right\} < 10^{-6}$.

14. In a square-shaped plaza, a metal wall is placed along one diagonal of the square. There are two metal poles placed on the two vertices not connected by the metal wall. A super-magnetic ball is uniformly randomly dropped in the square, and it pulls itself and sticks to the closest bit of metal. Consider the objects to have negligible thickness. The probability that the ball ends up stuck to the wall can be expressed as $\frac{1}{a}\left(b + c\sqrt{d}\right)$, where a is positive and minimal, d is not a multiple of a square number, and a, b, c, and d are integers. Find $a + b + c + d$.

15. Find the units digit of $\sum_{k=0}^{2022} \binom{2022}{k} \cos \frac{(1011-k)\pi}{2}$.