RAMC 2021

High School Individual Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. Using a standard 52-card deck of cards, a random set of 5 cards is selected. The probability that this set of cards will contain exactly three cards of the same rank can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. Find $a+b$.

   Answer: $\boxed{4259}$

   Solution: To construct a such a set of cards, first we select one of the 13 ranks, then select 3 of the 4 cards of that rank. Then, from the remaining 48 cards of a different rank, select 2 cards. This is given by $\binom{13}{1}\binom{4}{3}\binom{48}{2}$. There are $\binom{52}{5}$ total possible 5-card sets, so the desired probability is

$$\begin{aligned}\frac{\binom{13}{1}\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} &= \frac{13\cdot 4\cdot \frac{48!}{46!2!}}{\frac{52!}{47!5!}},\\ &= \frac{13\cdot 4\cdot 24\cdot 47}{13\cdot 17\cdot 10\cdot 49\cdot 24},\\ &= \frac{94}{4165}.\end{aligned}$$

   Thus, the answer is $94+4165=\boxed{4259}$.

2. For how many values of x does the expression $\sqrt{150-\sqrt{x}}$ evaluate to an integer?

   Answer: $\boxed{13}$

   Solution: Since we can always choose a value for x such that $\sqrt{x}$ is an integer on $[0,150]$ (by choosing x to be a square number), we are able to make the expression above equal to any non-negative integer less than $\sqrt{150}$. The question is essentially asking us to count the number of square numbers less than 150. The largest square number less than 150 is $12^2=144$, and including 0, there are a total of $\boxed{13}$ values.

3. Jacob writes N, a three-digit multiple of 7, on a whiteboard, then covers one of the digits with a piece of paper. Julie knows that N is a multiple of 7. No matter which of the three digits are being covered, if Julie looks at the board, she can be certain of the value of the digit that is being covered. How many possible values of N are there?

   Answer: $\boxed{12}$

   Solution: The only way that Julie can be sure of the value of the covered digit is if there is only one possible value for that digit. In order for this to be true, the digit must be 3, 4, 5, or 6, since for any other value, it is possible to add or subtract 7 from the digit and get another three-digit multiple of 7 without changing the values of the other digits. The exception is the hundreds place, which can be 7, since 0 cannot be the leading digit.

   The divisibility rule for 7 says that a three digit number $100a + 10b + c$ is divisible by 7 if and only if $10a + b - 2c$ is also divisible by 7. Since the values of a, b, and c are restricted, the minimum value of $10a + b - 2c$ is $10 \cdot 3 + 3 - 2 \cdot 6 = 21$ and the maximum is $10 \cdot 7 + 6 - 2 \cdot 3 = 70$. Thus, the possible values for $10a + b - 2c$ are 21, 28, 35, 42, 49, 56, 63, and 70. Also, $-9 \leq b - 2c \leq 0$ when b and c are 3, 4, 5, or 6, thus there is only one value of a for each value of $10a + b - 2c$. We work through the cases, looking for valid pairs of (b, c) for each value of $10a + b - 2c$ and a.

   For $10a + b - 2c = 21$, $a = 3 \implies b - 2c = -9$, and we have $(b, c) = (3, 6)$.
   For $10a + b - 2c = 28$, $a = 3 \implies b - 2c = -2$, and we have $(b, c) = (4, 3)$ and $(6, 4)$.
   For $10a + b - 2c = 35$, $a = 4 \implies b - 2c = -5$, and we have $(b, c) = (3, 4)$ and $(5, 5)$.
   For $10a + b - 2c = 42$, $a = 5 \implies b - 2c = -8$, and we have $(b, c) = (4, 6)$.
   For $10a + b - 2c = 49$, $a = 5 \implies b - 2c = -1$, and we have $(b, c) = (5, 3)$.
   For $10a + b - 2c = 56$, $a = 6 \implies b - 2c = -4$, and we have $(b, c) = (4, 4)$ and $(6, 5)$.
   For $10a + b - 2c = 63$, $a = 7 \implies b - 2c = -7$, and we have $(b, c) = (3, 5)$ and $(5, 6)$.
   For $10a + b - 2c = 70$, $a = 7 \implies b - 2c = 0$, and we have $(b, c) = (6, 3)$.

   There are $\boxed{12}$ numbers for which this is true: 336, 343, 364, 434, 455, 546, 553, 644, 665, 735, 756, and 763.

4. On triangle ABC, $AB = 4$, $AC = 6$, and $m\angle A = 120°$. The angle bisector of $\angle A$ intersects side AC at L. The length of segment AL can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. What is $a+b$?

   Answer: $\boxed{17}$

   Solution: Let perpendiculars from L to sides AB and AC meet these sides at E and F, respectively. Note that $\frac{1}{2}AB \cdot AC \cdot \sin 120° = 6\sqrt{3} = [\triangle ABC] = [\triangle ABL] + [\triangle ACL] = \frac{1}{2}AB \cdot EL + \frac{1}{2}AC \cdot FL$.
   Also, $\triangle AEL \cong \triangle AFL$ since they share side AL and have congruent angles, so $EL = FL$.
   Then, $6\sqrt{3} = \frac{1}{2} \cdot 4 \cdot EL + \frac{1}{2} \cdot 6 \cdot EL = 5 \cdot EL \implies EL = FL = \frac{6}{5}\sqrt{3}$, and since $\triangle AEL$ is a 30-60-90 triangle, $AL = \frac{2}{\sqrt{3}}EL = \frac{12}{5}$, and the answer is $12 + 5 = \boxed{17}$.

   Alternatively, using the Law of Cosines, we have $BC^2 = 4^2 + 6^2 - 2 \cdot 4 \cdot 6 \cdot \cos 120° \implies c = \sqrt{76}$.
   By the Angle Bisector Theorem, $\frac{BL}{CL} = \frac{AB}{AC} \implies BL = \frac{2}{5}\sqrt{76}$ and $CL = \frac{3}{5}\sqrt{76}$.
   Finally, using Stewart's Theorem,

$$\frac{2}{5}\sqrt{76} \cdot \frac{3}{5}\sqrt{76} \cdot \sqrt{76} + \sqrt{76}AL^2 = 4^2 \cdot \frac{3}{5}\sqrt{76} + 6^2 \cdot \frac{2}{5}\sqrt{76}.$$

   Simplifying yields $AL = \frac{12}{5}$, and the answer is $12 + 5 = \boxed{17}$.

5. Suppose $\sin\alpha + \sin\beta = \frac{\sqrt{6}}{3}$ and $\cos\alpha + \cos\beta = \frac{\sqrt{3}}{3}$. The value of $\cos^2\left(\frac{\alpha-\beta}{2}\right)$ can be expressed as a ratio of two relatively prime positive integers, $\frac{m}{n}$. Find $m+n$.

   Answer: $\boxed{5}$

   Solution: First, square both equations to obtain $\sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta = \frac{2}{3}$ and
   $\cos^2\alpha + 2\cos\alpha\cos\beta + \cos^2\beta = \frac{1}{3}$.
   Sum the equations and simplify:

$$\begin{aligned}\left(\sin^2\alpha + \cos^2\alpha\right) + 2\left(\sin\alpha\sin\beta + \cos\alpha\cos\beta\right) + \left(\sin^2\beta + \cos^2\beta\right) &= \frac{2}{3} + \frac{1}{3},\\ 1 + 2\cos(\alpha-\beta) + 1 &= 1,\\ \cos(\alpha-\beta) &= -\frac{1}{2}.\end{aligned}$$

   Then, $2\cos^2\left(\frac{\alpha-\beta}{2}\right) - 1 = \cos(\alpha-\beta) = -\frac{1}{2} \implies \cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{4}$ and thus the answer is $1 + 4 = \boxed{5}$.

6. Regular pentagon $ABCDE$ has side length 4. Triangle ABD is constructed, and its incircle is drawn, which is tangent to side AD and BD at points P and Q, respectively. The area of triangle PQD can be expressed as $a\sqrt{\frac{1}{2}(b-\sqrt{c})}$, where c is not a multiple of any square number, and a, b, and c are integers. Find $a+b+c$.

Answer: $\boxed{15}$

Solution: The ratio of the diagonals of a regular pentagon to its side length is the golden ratio (you can view a nice visual and accompanying proof here), or $\frac{1+\sqrt{5}}{2}$. Thus, $AD = BD = 2+2\sqrt{5}$.

The height from D to AB is, using the Pythagorean Theorem, $\sqrt{\left(2+2\sqrt{5}\right)^2 - 2^2} = 2\sqrt{5+2\sqrt{5}}$.

The area, A, of $\triangle ABD$ is $\frac{1}{2}\cdot 4\cdot 2\sqrt{5+2\sqrt{5}} = 4\sqrt{5+2\sqrt{5}}$.

The semiperimeter, s, of $\triangle ABD$ is $\frac{1}{2}\left(2\cdot\left(2+2\sqrt{5}\right)+4\right) = 4+2\sqrt{5}$.

The inradius is given by $i = \frac{A}{s} = \frac{4\sqrt{5+2\sqrt{5}}}{4+2\sqrt{5}} = 2\sqrt{5+2\sqrt{5}}\left(\sqrt{5}-2\right) = \frac{2\sqrt{5}\left(\sqrt{5}-2\right)}{\sqrt{5-2\sqrt{5}}} = 2\sqrt{5-2\sqrt{5}}$.

Let the center of the incircle be O and let the midpoint of AB be M. Then, $\triangle OPD \sim \triangle AMD$.

Thus, we have $\frac{OP}{AM} = \frac{DM}{DP}$.

Since OP is an inradius, we get $\frac{2\sqrt{5-2\sqrt{5}}}{2} = \frac{DM}{2\sqrt{5+2\sqrt{5}}} \implies DM = 2\sqrt{5}$.

Since the ratio of areas is the square of the ratio of side lengths, the area of $\triangle PQD$ is

$$
\begin{aligned}
[PQD] &= \left(\frac{2\sqrt{5}}{2+2\sqrt{5}}\right)^2\left(4\sqrt{5+2\sqrt{5}}\right), \\
&= \frac{5}{2}\left(3-\sqrt{5}\right)\sqrt{5+2\sqrt{5}}, \\
[PQD]^2 &= \frac{25}{2}\left(7-3\sqrt{5}\right)\left(5+2\sqrt{5}\right), \\
&= \frac{25}{2}\left(5-\sqrt{5}\right), \\
[PQD] &= 5\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}.
\end{aligned}
$$

Thus the answer is $5+5+5 = \boxed{15}$.

7. Roots r_1, r_2, and r_3 of the polynomial $x^3 - 5x^2 - 8x + a$ satisfy the equation $r_1 + 2r_2 + 4r_3 = 0$. What is the sum of the possible values of a?

Answer: $\boxed{60}$

Solution: Using Vieta's formulas, we obtain the following system of equations:

$$r_1 + r_2 + r_3 = 5 \tag{1}$$
$$r_1r_2 + r_2r_3 + r_3r_1 = -8 \tag{2}$$
$$r_1r_2r_3 = -a \tag{3}$$
$$r_1 + 2r_2 + 4r_3 = 0 \tag{4}$$

Then,

$$(4) - (1) \implies r_2 = -5 - 3r_3 \tag{5}$$
$$(4) - 2(1) \implies r_1 = 10 + 2r_3 \tag{6}$$

Substituting (5) and (6) into (2) gives

$$\begin{aligned}(10 + 2r_3)(-5 - 3r_3) + (-5 - 3r_3)r_3 + r_3(10 + 2r_3) &= -8,\\ r_3^2 + 5r_3 + 6 &= 0,\\ r_3 &= -2, -3.\end{aligned}$$

Substituting r_3 into (5), (6) and finally (3) yields $(r_1, r_2, r_3, a) = (6, 1, -2, 12), (4, 4, -3, 48)$. Thus, the answer is $12 + 48 = \boxed{60}$.

8. Let $S(n,\theta) = \prod_{i=0}^{n} \sin(2^i\theta)$ and similarly let $C(n,\theta) = \prod_{i=0}^{n} \cos(2^i\theta)$. For any integer N, there exists a pair of integers, (j,k), such that for all values of θ, $S(N-1,\theta) = 2^j \sin^k(\theta) \prod_{n=0}^{N-2} C(n,\theta)$. When simplified, $j+k = \frac{1}{2}P(N)$, where $P(N)$ is a polynomial of N. Determine the sum of the coefficients of $P(N)$.

Answer: $\boxed{2}$

Solution: Consider $C(n,\theta)$. We have

$$\begin{aligned}
C(n,\theta) &= \cos(\theta)\cos(2\theta)\cos(2^2\theta)\cdots\cos(2^n\theta),\\
\sin(\theta)\cdot C(n,\theta) &= \sin(\theta)\cos(\theta)\cos(2\theta)\cos(2^2\theta)\cdots\cos(2^n\theta),\\
\sin(\theta)\cdot C(n,\theta) &= \frac{1}{2}\sin(2\theta)\cos(2\theta)\cos(2^2\theta)\cdots\cos(2^n\theta),\\
\sin(\theta)\cdot C(n,\theta) &= \frac{1}{2^2}\sin(2^2\theta)\cos(2^2\theta)\cdots\cos(2^n\theta),\\
&\vdots\\
\sin(\theta)\cdot C(n,\theta) &= \frac{1}{2^{n+1}}\sin(2^{n+1}\theta),\\
C(n,\theta) &= \frac{1}{2^{n+1}\sin(\theta)}\sin(2^{n+1}\theta).
\end{aligned}$$

Now,

$$\begin{aligned}
\prod_{n=0}^{N-2} C(n,\theta) &= \frac{1}{2^{1+2+\cdots+N-1}\sin^{N-1}(\theta)}\sin(2\theta)\sin(2^2\theta)\cdots\sin(2^{N-1}\theta),\\
&= \frac{1}{2^{\frac{(1+N-1)(N-1)}{2}}\sin^{N}(\theta)}\sin(\theta)\sin(2\theta)\sin(2^2\theta)\cdots\sin(2^{N-1}\theta),\\
&= \frac{1}{2^{\frac{(N)(N-1)}{2}}\sin^{N}(\theta)}S(N-1,\theta).
\end{aligned}$$

Thus, $j+k = \frac{(N)(N-1)}{2} + N = \frac{N^2+N}{2}$, and the answer is $1+1 = \boxed{2}$.

9. George labels 1000 quarters with the numbers 1 to 1000. He lays them all on a table, heads facing up. He flips over each coin whose label is a multiple of 3 but not 5. Then, he flips over each coin whose label is a multiple of 5 but not 7. Finally, he flips over each coin whose label is a multiple of 7 but not 11. George now chooses two unique coins on the table. The probability that both of the chosen coins are tails facing up can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. What is $a+b$?

Answer: $\boxed{833}$

Solution: The number of integers from 1 to 1000 that are in sets A, B, and C are as follows:

$$A = \{\text{multiples of 3 but not 5}\}, \quad |A| = \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor = 267,$$
$$B = \{\text{multiples of 5 but not 7}\}, \quad |B| = \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{35} \right\rfloor = 172,$$
$$C = \{\text{multiples of 7 but not 11}\}, \quad |C| = \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{77} \right\rfloor = 130.$$

Since B contains only multiples of 5 and A contains no multiples of 5, $A \cap B = \varnothing$, and similarly, $B \cap C = \varnothing$, and $A \cap B \cap C = \varnothing$. However, $A \cap C$ contains the multiples of 3 and 7 but not 5 or 11, so

$$|A \cap C| = \left\lfloor \frac{1000}{21} \right\rfloor - \left\lfloor \frac{1000}{105} \right\rfloor - \left\lfloor \frac{1000}{232} \right\rfloor + \left\lfloor \frac{1000}{1155} \right\rfloor = 47 - 9 - 4 + 0 = 34.$$

Thus, using the Principle of Inclusion and Exclusion, and considering that coins flipped twice return to heads, the number of tails is $[267 + 172 + 130 - (0 + 0 + 34) + (0)] - 34 = 501$.
So, the probability of selecting two tails coins is $\frac{501}{1000} \cdot \frac{500}{999} = \frac{167}{666}$ and the answer is $167 + 666 = \boxed{833}$.

10. How many ordered pairs of integers, (x, y), satisfy the equation $\frac{1}{x^2} - \frac{3}{y} = \frac{1}{25}$?

Answer: $\boxed{4}$

Solution: Multiplying both sides by $25x^2y$ and moving everything to one side yields $x^2y - 25y + 75x^2 = 0$, or $(x-5)(x+5)(y+75) = -1875 = -3 \cdot 5^4$. The possible values for these three terms are the positive and negative divisors of -1875, which are 1, 3, 5, 15, 25, 75, 125, 375, 625, 1875, and each corresponding negative value. In order for the first two terms to differ by 10, they must be one of the following: (-25)(-15), (-15)(-5), (-5)(5), (5)(15), or (15)(25). Thus, there are $\boxed{5}$ solutions to the equation.

11. Find the sum of the values of x for which $\log_2 x \log_3 x \log_4 x = \log_2 x(\log_2 x + \log_3 x + \log_4 x)$.

Answer: $\boxed{109}$

Solution: Moving everything to the left hand side and expanding gives

$$\log_2 x \log_3 x \log_4 x - \log_2 x \log_2 x - \log_2 x \log_3 x - \log_2 x \log_4 x = 0.$$

Changing bases to the natural logarithm and multiplying by $\ln^2 2 \ln 3 \ln 4$ gives

$$\ln^3 x \ln 2 - \ln^2 x \ln 3 \ln 4 - \ln^2 x \ln 2 \ln 4 - \ln^2 x \ln 2 \ln 3 = 0.$$

Factoring out $\ln^2 x$ gives

$$\ln^2 x(\ln x \ln 2 - \ln 3 \ln 4 - \ln 2 \ln 4 - \ln 2 \ln 3) = 0.$$

Now, either $\ln^2 x = 0 \implies x = 1$, or

$$\begin{aligned} 0 &= \ln x \ln 2 - \ln 3 \ln 4 - \ln 2 \ln 4 - \ln 2 \ln 3, \\ \ln x \ln 2 &= 2 \cdot \ln 3 \ln 2 + \ln 2 \ln 4 + \ln 2 \ln 3, \\ \ln x &= 2 \ln 3 + \ln 4 + \ln 3, \\ e^{\ln x} &= e^{2 \ln 3 + \ln 4 + \ln 3}, \\ x &= 9 \cdot 4 \cdot 3, \\ x &= 108. \end{aligned}$$

Thus, the answer is $1 + 108 = \boxed{109}$.

12. How many strictly increasing three-term geometric sequences with integer ratios can be formed by using only the positive integers between 1 and 169, inclusive?

Answer: $\boxed{91}$

Solution: The common ratio r can be between 2 and 13, inclusive.

When $10 \leq r \leq 13$, the first term must be 1.

When $r = 8$ or $r = 9$, the first term can be either 1 or 2

When $r = 7$, the first term can be between 1 and 3, inclusive.

When $r = 6$, the first term can be between 1 and 4, inclusive.

When $r = 5$, the first term can be between 1 and 6, inclusive.

When $r = 4$, the first term can be between 1 and 10, inclusive.

When $r = 3$, the first term can be between 1 and 18, inclusive.

When $r = 2$, the first term can be between 1 and 42, inclusive.

Thus, there are $4 + 2 \cdot 2 + 3 + 4 + 6 + 10 + 18 + 42 = \boxed{91}$ such sequences.

Note: The original problem statement neglected to specify that the sequence required integer ratios. Without this requirement, the answer is $\boxed{198}$.

13. For any positive real number x, let $[x]$ denote the greatest integer no larger than x, and let $\{x\}$ denote the decimal portion of x, such that $[x] + \{x\} = x$. Find the smallest possible positive integer n such that $1 - \left[\sqrt{n^2+1} + n\right] \cdot \left\{\sqrt{n^2+1} + n\right\} < 10^{-6}$.

Answer: $\boxed{500}$

Solution: Since $n > 0$, we know that $n < \sqrt{n^2+1} < n + 1$, so $[\sqrt{n^2+1} + n] = 2n$ and thus $\{\sqrt{n^2+1} + n\} = \left(\sqrt{n^2+1} + n\right) - 2n = \sqrt{n^2+1} - n$.

Then, $1 - 2n\left(\sqrt{n^2+1} - n\right) < 10^{-6}$, so $1 - 10^{-6} < \dfrac{2n}{\sqrt{n^2+1} + n}$.

Now let $m = \dfrac{1}{\sqrt{n^2+1} + n}$. Then, $n = \dfrac{1 - m^2}{2m}$.

Substituting this back in, we have $2 \cdot \dfrac{1 - m^2}{2m} \cdot m > 1 - 10^{-6} \implies m < 10^{-3}$.

Thus, $n > \dfrac{1 - \left(10^{-3}\right)^2}{2 \cdot 10^{-3}} = \dfrac{999999}{2000}$, and the least integer value of n is $\boxed{500}$.

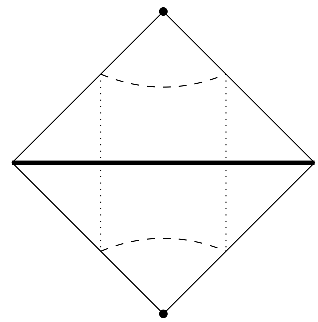
14. In a square-shaped plaza, a metal wall is placed along one diagonal of the square. There are two metal poles placed on the two vertices not connected by the metal wall. A super-magnetic ball is uniformly randomly dropped in the square, and it pulls itself and sticks to the closest bit of metal. Consider the objects to have negligible thickness. The probability that the ball ends up stuck to the wall can be expressed as $\frac{1}{a}\left(b+c\sqrt{d}\right)$, where a is positive and minimal, d is not a multiple of a square number, and a, b, c, and d are integers. Find $a+b+c+d$.

Answer: $\boxed{37}$

Solution: See the diagram to the right.

The key here is that the boundary between the ball sticking to the wall and a pole is when the ball is equidistant from the poles and the wall, which is also the definition of a parabola. In the diagram, the dashed lines are these boundary parabolas.

Since we are trying to find the probability that a randomly dropped ball sticks to the wall, we need to find the ratio of the area between the parabolas to the whole square. To do this, we will analyze the square in the coordinate plane.



We choose the vertices of the square to be $(\pm 2, 0)$ and $(0, \pm 2)$. Then, the equations for the two parabolas are $p_1(x) = \frac{1}{4}x^2 + 1$ and $p_2(x) = -\frac{1}{4}x^2 - 1$.

Intersecting the two parabolas and the square's perimeter yields: $\left(\pm\left[2\sqrt{2}-2\right], \pm\left[4-2\sqrt{2}\right]\right)$.

The two triangles bounded by the dotted lines have a total area of $\frac{1}{2}(4-2\sqrt{2}-(2\sqrt{2}-4))^2 = 48-32\sqrt{2}$.

The area bounded by the dotted and dashed lines is given by

$$\int_{2-2\sqrt{2}}^{2\sqrt{2}-2} \frac{1}{4}x^2 + 1 - \left(-\frac{1}{4}x^2 - 1\right) dx = \frac{16}{3}\left(4\sqrt{2}-5\right).$$

Adding the areas together gives $48 - 32\sqrt{2} + \frac{16}{3}\left(4\sqrt{2}-5\right) = \frac{1}{3}\left(64-32\sqrt{2}\right)$, and so the answer is $3 + 64 - 32 + 2 = \boxed{37}$.

15. Find the units digit of $\displaystyle\sum_{k=0}^{2022}\binom{2022}{k}\cos\frac{(1011-k)\pi}{2}$.

Answer: $\boxed{8}$

Solution: When k is even, $\cos\dfrac{(1011-k)\pi}{2}=0$, and thus do not contribute to the sum, so we only need to consider odd values of k. Let n be a non-negative integer. When $k=4n+1$, $\cos\dfrac{(1011-k)\pi}{2}=-1$ and when $k=4n+3$, $\cos\dfrac{(1011-k)\pi}{2}=1$. Now, we have

$$\sum_{k=0}^{2022}\binom{2022}{k}\cos\frac{(1011-k)\pi}{2}=-\binom{2022}{1}+\binom{2022}{3}-\binom{2022}{5}+\cdots-\cdots+\binom{2022}{2019}-\binom{2022}{2021}.$$

Next, consider the complex expression $i(1+i)^{2022}$. The real part of this expression is

$$\begin{aligned}\operatorname{Re}\left[i(1+i)^{2022}\right]&=i\left(\binom{2022}{1}i^{2021}+\binom{2022}{3}i^{2019}+\cdots+\binom{2022}{2019}i^{3}+\binom{2022}{2021}i^{1}\right)\\&=i\left(\binom{2022}{1}i^{1}+\binom{2022}{3}i^{3}+\cdots+\binom{2022}{2019}i^{3}+\binom{2022}{2021}i^{1}\right)\\&=-\binom{2022}{1}+\binom{2022}{3}+\cdots-\cdots+\binom{2022}{2019}-\binom{2022}{2021},\\&=\sum_{k=0}^{2022}\binom{2022}{k}\cos\frac{(1011-k)\pi}{2}.\end{aligned}$$

Finally, $i(1+i)^{2022}=i[(1+i)^2]^{1011}=i(2i)^{1011}=2^{1011}i^{1012}=2^{1011}$. Notice that is is purely real, so $\displaystyle\sum_{k=0}^{2022}\binom{2022}{k}\cos\frac{(1011-k)\pi}{2}=\operatorname{Re}\left[i(1+i)^{2022}\right]=2^{1011}$.

Looking at the first couple powers of 2, we have $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$. The units digits repeat with a period of 4, and since 1012 is a multiple of 4, the units digit of 2^{1011} must be $\boxed{8}$.