RAMC 2021

High School Team Round

- **SCORING:** The first 5 questions are worth 2 points each, and last 5 questions are worth 3 points each.
- This round contains 10 questions to be solved in 25 minutes. Problems towards the end tend to be more difficult than problems toward the beginning.
- No computational aids are permitted other than scratch paper, graph paper, and a pen/pencil. No calculators of any kind are allowed.
- All answers are integers. When submitting answers, do not add additional characters (such as spaces or units) beyond pure numerical digits, with the exception of a minus (-) symbol when needed.
- If you believe there is an error on the test, submit a challenge to rochestermathclub@gmail.com. Please include your name, level (Elem I/II, MS, HS), and explanation of the problem and your solution.

Take a moment to check that your information is entered correctly!

1. When the 4 solutions to $z^4 + z^3 + z^2 - z = 2$ are graphed on the complex plane and adjacent points are connected, a convex quadrilateral is formed. Let p denote the perimeter of this quadrilateral and q denote the area of this quadrilateral. Then, pq can be expressed in the form $a\sqrt{b} + c\sqrt{d}$, where neither b nor d are a multiple of any square number, and a, b, c, and d are integers. Find $a + b + c + d$.

2. How many non-empty subsets of $\{1, 2, \cdots, 10\}$ exist such that none of the elements in the subset are consecutive? For example, $\{1, 4, 6\}$ is a valid subset, but $\{1, 4, 5\}$ is not.

3. For some value of θ, the following system of equations has no real solutions:

$$\frac{x}{2} + y\cos\theta = \sqrt{2}$$
$$\frac{y}{2} + z\sin\theta = -1$$
$$z + w\cos\theta = -\sqrt{3}$$
$$w + x\sin\theta = 2$$

   Compute $\tan^2\theta$.

4. Find the smallest possible value for a positive integer n such that $7^n + 2n$ is divisible by 57.

5. A circle with radius 4 is centered at the origin. For any right triangle with legs perpendicular to the coordinate axes inscribed within this circle, a segment is constructed from its incenter to the origin, and the midpoint of this segment is labeled M. The locus of all possible points M forms the boundary of a region whose area can be expressed as $a + b\pi$, where a and b are integers. What is $a + b$?

6. Four positive real numbers, a, b, c, and d, are chosen such that $(a + c)(b + d) = ac + bd$. Find the smallest possible value of $S = \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$.

7. The floor function $\lfloor x \rfloor$ returns the greatest integer no larger than x. There is a sequence $\{a_n\}$ such that $a_n = \left\lfloor (2 + \sqrt{3})^{2^n} \right\rfloor$ for any positive integer n. Find the two-digit number that appears in the last two digits of a_{2021}.

8. Find the number of integer triples (x, y, z) that satisfy all the following conditions:

$$x \neq y, y \neq z, z \neq x$$

$$1 \leq x, y, z \leq 100$$

$$x + y = 3z + 10$$

9. Suppose m is a positive real number, and the system of equations

$$\sin x = m\cos^3 y$$
$$\cos x = m\sin^3 y$$

has real number solution(s) for (x, y). The value of m is bounded above by a, and below by b. Find $a + b$.

10. Equilateral triangle ABC has side length 1. Points D and E are on sides AB and AC, respectively, such that when triangle ADE is folded along the crease DE, point A lies on side BC. The minimum possible length of AD can be expressed as $a\sqrt{b} + c$, where b is not a multiple of any square number, and a, b, and c are integers. Find $a + b + c$.