RAMC 2021

High School Team Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. When the 4 solutions to $z^4 + z^3 + z^2 - z = 2$ are graphed on the complex plane and adjacent points are connected, a convex quadrilateral is formed. Let p denote the perimeter of this quadrilateral and q denote the area of this quadrilateral. Then, pq can be expressed in the form $a\sqrt{b} + c\sqrt{d}$, where neither b nor d are a multiple of any square number, and a, b, c, and d are integers. Find $a + b + c + d$.

   Answer: $\boxed{27}$

   Solution: Moving the 2 to the left hand side and factoring gives $(z-1)(z+1)(z^2+z+2) = 0$. The four solutions are thus ± 1 and $-\frac{1}{2} \pm \frac{i\sqrt{7}}{2}$.
   The distance from 1 to $-\frac{1}{2} + \frac{i\sqrt{7}}{2}$ is $\sqrt{\left(-\frac{1}{2} - 1\right)^2 + \left(0 - \frac{\sqrt{7}}{2}\right)^2} = 2$.
   The distance from -1 to $-\frac{1}{2} + \frac{i\sqrt{7}}{2}$ is $\sqrt{\left(-\frac{1}{2} - (-1)\right)^2 + \left(0 - \frac{\sqrt{7}}{2}\right)^2} = \sqrt{2}$.
   By symmetry, $p = 2\left(2 + \sqrt{2}\right) = 4 + 2\sqrt{2}$.
   Since the solutions form a kite in the complex plane, $q = \frac{1}{2}(1 - (-1))\left(\frac{\sqrt{7}}{2} - \left(-\frac{\sqrt{7}}{2}\right)\right) = \sqrt{7}$.
   Thus, $pq = \left(4 + 2\sqrt{2}\right)\sqrt{7} = 4\sqrt{7} + 2\sqrt{14}$, and the answer is $4 + 7 + 2 + 14 = \boxed{27}$.

   Note: The original problem asked for $a+b+c$ where the answer was in the form $a\sqrt{b}+\sqrt{c}$ because the original solution incorrectly dropped a coefficient, concluding that $pq = \left(4 + \sqrt{2}\right)\sqrt{7} = 4\sqrt{7}+\sqrt{14} \implies a+b+c = 4+7+14 = \boxed{25}$. Since the correct value is $pq = 4\sqrt{7}+2\sqrt{14} = 4\sqrt{7}+\sqrt{56} = 2\sqrt{14}+\sqrt{112}$, the answers $4 + 7 + 56 = \boxed{67}$ and $2 + 14 + 112 = \boxed{128}$ were also accepted on competition day.

2. How many non-empty subsets of $\{1, 2, \cdots, 10\}$ exist such that none of the elements in the subset are consecutive? For example, $\{1, 4, 6\}$ is a valid subset, but $\{1, 4, 5\}$ is not.

   Answer: $\boxed{143}$

   Solution: Firstly there are 10 subsets that contain only one element, which are valid.
   For each of these subsets that do not contain 9 or 10, we can add another element that is at least 2 greater than the existing element. Thus there are $\sum_{n=1}^{8} n = 36$ valid 2-element subsets.
   For each of these 2-element subsets that do not contain 9 or 10, we can add another element that is at least 2 greater than the existing elements. Thus there are $\sum_{n=1}^{6}\sum_{m=1}^{n} m = 56$ valid 3-element subsets.
   Continuing this pattern, there are $\sum_{n=1}^{4}\sum_{m=1}^{n}\sum_{p=1}^{m} p = 35$ valid 4-element subsets,
   and $\sum_{n=1}^{2}\sum_{m=1}^{n}\sum_{p=1}^{m}\sum_{q=1}^{p} q = 6$ valid 5-element subsets.
   In total, there are $10 + 36 + 56 + 35 + 6 = \boxed{143}$ valid subsets.

3. For some value of θ, the following system of equations has no real solutions:

$$\begin{aligned} \frac{x}{2} + y\cos\theta &= \sqrt{2} \\ \frac{y}{2} + z\sin\theta &= -1 \\ z + w\cos\theta &= -\sqrt{3} \\ w + x\sin\theta &= 2 \end{aligned}$$

Compute $\tan^2\theta$.

Answer: $\boxed{1}$

Solution: A system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ (where $\mathbf{A}$ is the coefficient matrix, $\vec{x}$ is a vector of variables, and $\vec{b}$ is a vector of constants) has no solutions if $\det(\mathbf{A}) = 0$. Organizing the given system in this fashion, we get the following determinant for the coefficient matrix:

$$\begin{aligned} \begin{vmatrix} \frac{1}{2} & \cos\theta & 0 & 0 \\ 0 & \frac{1}{2} & \sin\theta & 0 \\ 0 & 0 & 1 & \cos\theta \\ \sin\theta & 0 & 0 & 1 \end{vmatrix} &= \frac{1}{2}\begin{vmatrix} \frac{1}{2} & \sin\theta & 0 \\ 0 & 1 & \cos\theta \\ 0 & 0 & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} \cos\theta & 0 & 0 \\ \frac{1}{2} & \sin\theta & 0 \\ 0 & 1 & \cos\theta \end{vmatrix}, \\ &= \frac{1}{2}\cdot\frac{1}{2}\begin{vmatrix} 1 & \cos\theta \\ 0 & 1 \end{vmatrix} - \sin\theta\cdot\cos\theta \begin{vmatrix} \sin\theta & 0 \\ 1 & \cos\theta \end{vmatrix}, \\ &= \frac{1}{4} - \sin^2\theta\cos^2\theta. \end{aligned}$$

Since the determinant is 0, we have that

$$\begin{aligned} \sin^2\theta\cos^2\theta &= \frac{1}{4}, \\ 2\sin\theta\cos\theta\cdot 2\sin\theta\cos\theta &= 1, \\ \sin^2 2\theta &= 1, \\ \sin 2\theta &= \pm 1. \end{aligned}$$

Thus, $2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4} \implies \tan\theta = 1$. So, $\tan^2\theta = \boxed{1}$.

4. Find the smallest possible value for a positive integer n such that $7^n + 2n$ is divisible by 57.

Answer: $\boxed{25}$

Solution: Notice that $7^3 \equiv 1 \pmod{57}$. We will split into three cases.
Case 1: Let $n = 3k$ where k is a positive integer.
Then, $7^n = 7^{3k} \equiv 1 \pmod{57}$.
This implies $7^n + 2n \equiv 6k + 1 \pmod{57}$.
However, $6k + 1$ cannot be a multiple of 3 while 57 is, so this case is impossible.

Case 2: Let $n = 3k + 1$ where k is a positive integer.
Then, $7^n = 7^{3k+1} \equiv 7 \pmod{57}$.
Thus, $7^n + 2n \equiv 6k + 9 \pmod{57}$.
Solving for divisibility, we have $6k + 9 \equiv 0 \pmod{57} \implies 2k + 3 \equiv 0 \pmod{19}$.
Minimizing k yields $k = 8$ and $n = 25$.

Case 3: Let $n = 3k + 2$ where k is a positive integer.
Then, $7^n = 7^{3k+2} \equiv 49 \pmod{57}$.
This implies $7^n + 2n \equiv 6k + 53 \equiv 6k - 4 \pmod{57}$.
Similar to case 1, $6k - 4$ cannot be a multiple of 3 while 57 is, so this case is impossible.

Thus, the smallest possible value for n is $\boxed{25}$.

5. A circle with radius 4 is centered at the origin. For any right triangle with legs perpendicular to the coordinate axes inscribed within this circle, a segment is constructed from its incenter to the origin, and the midpoint of this segment is labeled M. The locus of all possible points M forms the boundary of a region whose area can be expressed as $a + b\pi$, where a and b are integers. What is $a + b$?

Answer: $\boxed{12}$

Solution: Let point (p, q) lie on the circle with radius 4. Then, an inscribed right triangle will have its other vertices at $(p, -q)$ and $(-p, q)$.

We can parameterize the circle and thus restrict (p, q) to only points on the circle, by letting $(p, q) = (4\cos\theta, 4\sin\theta)$ for $\theta \in [0, 2\pi]$. However, if (p, q) lies exclusively in the first quadrant, the shape of the right triangles constructed are the same as in the second, third, and fourth quadrants, just rotated $90°$ each time, and so the locus of the incenter (and thus the locus of the midpoint of the segments between the incenter and origin) will be the same shape, also just rotated $90°$ each time. We will focus on $p > 0$ and $q > 0$, or $\theta \in \left[0, \frac{\pi}{2}\right]$, and use symmetry to our advantage.

Notice that, since the legs of the triangles are parallel to the coordinate axes, the coordinates of the incenter can be expressed as $(p - i, q - i)$, where i is the inradius.

The inradius is given by $i = \frac{\text{area}}{\text{semiperimeter}} = \frac{2pq}{p + q + \sqrt{p^2 + q^2}} = p + q - \sqrt{p^2 + q^2}$.

Substituting for our parameterization, $i = 4\cos\theta + 4\sin\theta - 4$, and the coordinates of the incenter must be $(p - i, q - i) = (4 - 4\sin\theta, 4 - 4\cos\theta)$, for $\theta \in \left[0, \frac{\pi}{2}\right]$.

The coordinates of the midpoint of the segment between the incenter and the origin must be $\frac{1}{2}(4 - 4\sin\theta, 4 - 4\cos\theta) = (2 - 2\sin\theta, 2 - 2\cos\theta)$, for $\theta \in \left[0, \frac{\pi}{2}\right]$.

Notice that this is the parameterization for an arc of radius 2 centered at (2,2) that goes from (2,0) to (0,2) (verify by plugging the two coordinates into $(x - 2)^2 + (y - 2)^2 = 2^2$).

If we rotate and copy this arc $90°$ about the origin thrice, we obtain the locus of all the midpoints.

The area of this region can be found by connecting all of the centers of the arcs to form a square of length 4, and removing a quarter-circle of radius 2 from each corner.

The area is thus $4^2 - 4 \cdot \frac{1}{4} \cdot \left(2^2\pi\right) = 16 - 4\pi$ and the answer is $16 - 4 = \boxed{12}$.

6. Four positive real numbers, a, b, c, and d, are chosen such that $(a+c)(b+d) = ac+bd$. Find the smallest possible value of $S = \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$.

Answer: $\boxed{8}$

Solution: The AM-GM inequality states that $\frac{x+y}{2} \geq \sqrt{xy}$, and subsequently, $x+y \geq 2\sqrt{xy}$, when x and y are non-negative.
Thus, since a, b, c, and d are positive and $ac+bd = (a+c)(b+d)$,

$$\begin{aligned}
S &= \left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right), \\
&\geq 2\sqrt{\frac{ac}{bd}} + 2\sqrt{\frac{bd}{ac}}, \\
&= \frac{2(ac+bd)}{\sqrt{abcd}}, \\
&= \frac{2(a+c)(b+d)}{\sqrt{abcd}}, \\
&\geq \frac{2(2\sqrt{ac})(2\sqrt{bd})}{\sqrt{abcd}}, \\
&= 8.
\end{aligned}$$

So, $S \geq 8$ and thus its minimum value is $\boxed{8}$, which holds when $a = c = 2+\sqrt{3}$ and $b = d = 1$.

7. The floor function $\lfloor x \rfloor$ returns the greatest integer no larger than x. There is a sequence $\{a_n\}$ such that $a_n = \left\lfloor (2+\sqrt{3})^{2^n} \right\rfloor$ for any positive integer n. Find the two-digit number that appears in the last two digits of a_{2021}.

Answer: $\boxed{13}$

Solution: Let $a = 2+\sqrt{3}$ and $b = 2-\sqrt{3}$, thus $a+b = 4$ and $ab = 1$.
Then, let $b_n = a^{2^n} + b^{2^n}$. Looking at the first couple of terms,

$$\begin{aligned}
a^2+b^2 &= (a+b)^2 - 2ab = 14. \\
a^4+b^4 &= (a^2+b^2)^2 - 2(ab)^2 = 14^2 - 2 = 194. \\
a^8+b^8 &= (a^4+b^4)^2 - 2(ab)^4 = 194^2 - 2 = 37634.
\end{aligned}$$

In this sequence, the last two digits of the next term only depend on the last two digits of the previous term.
The last two digits of the first several terms of b_n are: $b_1 = 14$, $b_2 = \ldots 94$, $b_3 = \ldots 34$, $b_4 = \ldots 54$, $b_5 = \ldots 14$, $b_6 = \ldots 94$, $b_7 = \ldots 34$, $b_8 = \ldots 54$, $\cdots$
Since the sequence repeats with period 4, $b_{2021} = b_1 = \ldots 14$.
Notice that $0 < 2-\sqrt{3} < 1 \implies 0 < \left(2-\sqrt{3}\right)^{2^n} < 1$, and since $a_{2021} = b_{2021} - \left(2-\sqrt{3}\right)^{2^{2021}}$, the last two digits of a_{2021} must be $\boxed{13}$.

8. Find the number of integer triples (x, y, z) that satisfy all the following conditions:

$$x \neq y, y \neq z, z \neq x$$

$$1 \leq x, y, z \leq 100$$

$$x + y = 3z + 10$$

Answer: $\boxed{3194}$

Solution: We have $x + y = 3z + 10 \leq 200$, so $z \leq \frac{190}{3}$, thus the largest value of z is 63.

When $z = 63$, $x + y = 199$, x can be either 99 or 100 and $y = 199 - x$, so there are 2 triples.

When $z = 62$, $x + y = 196$, x can be between 96 and 100, inclusive, for a total of 5 triples.

When $z = 61$, $x + y = 193$, x can be between 93 and 100, inclusive, for a total of 8 triples.

...

When $z = 31$, $x + y = 103$, x can be between 3 and 100, inclusive, for a total of 98 triples.

Below $z = 31$, the number of triples follows a different pattern.

When $z = 30$, $x + y = 100$, x can be between 1 and 99, inclusive, for a total of 99 triples.

When $z = 29$, $x + y = 97$, x can be between 1 and 96, inclusive, for a total of 96 triples.

...

When $z = 1$, $x + y = 13$, x can be between 1 and 12, inclusive, for a total of 12 triples.

Now we have the sum $S = (2 + 5 + 8 + 11 + \cdots + 98) + (99 + 96 + 93 + \cdots + 12)$

$= (2 + 98) \cdot 33 \div 2 + (99 + 12) \cdot 30 \div 2 = 1650 + 1665 = 3315$ triples.

But remember x, y, and z are distinct so we need to remove triples which contain the same integer more than once.

When $x = y$, we have $2x = 3z + 10$, so when z is even and between 2 and 62, inclusive, we need to remove 1 triple. In total, 31 triples need to be removed.

When $x = z$, we have $y = 2z + 10$, so when z is between 1 and 45, inclusive (here y is still from 1 to 100), 1 triple needs to be removed. In total, 45 triples need to be removed.

The $y = z$ case is symmetric to $x = z$ case, so another 45 triples need to be removed.

We don't need to consider $x = y = z$ because then the right side would be larger than the left side and equation does not hold.

Thus, the answer is $3315 - 31 - 45 \cdot 2 = \boxed{3194}$ triples.

9. Suppose m is a positive real number, and the system of equations

$$\sin x = m\cos^3 y$$
$$\cos x = m\sin^3 y$$

has real number solution(s) for (x, y). The value of m is bounded above by a, and below by b. Find $a + b$.

Answer: $\boxed{3}$

Solution: Square both equations and add them together to get

$$\begin{aligned}
1 &= m^2\left(\sin^6 y + \cos^6 y\right),\\
1 &= m^2\left(\sin^2 y + \cos^2 y\right)\left(\sin^4 y - \sin^2 y\cos^2 y + \cos^4 y\right),\\
1 &= m^2\left(\left(\sin^2 y + \cos^2 y\right)^2 - 3\sin^2 y\cos^2 y\right),\\
m^2 &= \frac{1}{1 - 3(\sin y\cos y)^2} = \frac{1}{1 - \frac{3}{4}\sin^2(2y)}.
\end{aligned}$$

We know $-1 \le \sin(2y) \le 1 \implies 0 \le \sin^2(2y) \le 1$, thus $1 \le m^2 \le 4$. Since m is positive, $1 \le m \le 2$, and the answer is $1 + 2 = \boxed{3}$.

10. Equilateral triangle ABC has side length 1. Points D and E are on sides AB and AC, respectively, such that when triangle ADE is folded along the crease DE, point A lies on side BC. The minimum possible length of AD can be expressed as $a\sqrt{b} + c$, where b is not a multiple of any square number, and a, b, and c are integers. Find $a + b + c$.

Answer: $\boxed{2}$

Solution: Let A' be the point where A lies after being folded along DE. Since $DA = DA'$, if we draw a circle with radius DA centered at D, no matter where E is, A' will always lie on this circle. Thus, the smallest possible value of AD is achieved when this circle is tangent to BC, so that A' can just barely lie on BC.

Point D is the center of the circle, and since the circle is tangent to BC, $DA' \perp BC$, so $\triangle DBA'$ must be a 30-60-90 triangle, thus $DA' = \frac{\sqrt{3}}{2}BD = DA$.

We know $BD + DA = 1$, thus $\frac{2}{\sqrt{3}}DA + DA = 1 \implies DA = 2\sqrt{3} - 3$, and thus the answer is $2 + 3 - 3 = \boxed{2}$.