RAMC 2021

Middle School Individual Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. Johnny needs to buy 22 sodas. The store sells sodas in packs of 10, 5, and 1. The 10-pack costs \$15.20, the 5-pack costs \$7.30, and the 1-pack costs \$1.50. If Johnny wants to spend the least amount of money to , how many packs will he buy?

   Answer: $\boxed{6}$

   Solution: First we solve for the unit cost of a bottle of soda from each pack. For the 10-pack, each bottle costs \$15.20 $\div$ 10 = \$1.52 per bottle. For the 5-pack, each bottle costs \$7.30 $\div$ 5 = \$1.46 per bottle. The 1-pack, of course, costs \$1.50 per bottle. To minimize the cost, we buy four 5-packs, since it has the least unit cost, then two 1-packs, the next least unit cost, for a total of $\boxed{6}$ packs.

2. Let $a\Omega b = a^2 - 2ab$ and let $a\Phi b = b^2 - 2a$. What is the value of $((25\Omega 2) - (3\Phi 5))\Phi 2$?

   Answer: $\boxed{-1008}$

   Solution: We treat Ω and Φ as operations, so

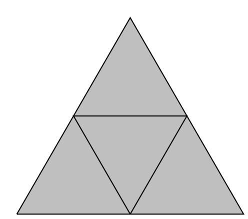
$$\begin{aligned}((25\Omega 2) - (3\Phi 5))\Phi 2 &= ((25^2 - 2\cdot 25\cdot 2) - (5^2 - 2\cdot 3))\Phi 2,\\ &= (525 - 19)\Phi 2,\\ &= 506\Phi 2,\\ &= 2^2 - 2\cdot 506,\\ &= \boxed{-1008}.\end{aligned}$$

3. There are 120 students at a graduation ceremony. It takes 5 seconds for each student to walk across the stage. As the students begin walking across the stage, Mr. Lee's digital watch reads 01:00 PM. What is the sum of the digits on Mr. Lee's watch after the last student walks across the stage?

   Answer: $\boxed{2}$

   Solution: The total time it takes 120 students to walk across the stage is $120 \cdot 5 = 600$ seconds, or 10 minutes. Thus, Mr. Lee's watch will display 01:10 PM, so the answer is $0 + 1 + 1 + 0 = \boxed{2}$.

4. The figure to the right consists of four equilateral triangles of side length s. The height of the entire figure is 24. The value s can be expressed as $a\sqrt{b}$, where a is an integer and b is not a multiple of any perfect square larger than 1. Find $a + b$.



   Answer: $\boxed{11}$

   Solution: The height of the figure is 24, so we know the height of one of the small triangles is 12. Drawing the height of an equilateral triangle splits it into two 30-60-90 triangles, thus the side length of a small triangle is $2 \cdot \frac{12}{\sqrt{3}}$. Simplifying this yields $8\sqrt{3}$ and thus the answer is $8 + 3 = \boxed{11}$.

5. A car's instantaneous velocity, in meters per second, is given by the function $f(x) = x^2 + 5$, where x is the number of seconds after the car begins moving. What is the car's instantaneous velocity in *kilometers per second* when 2 minutes have passed? Round your answer to the nearest integer.

   Answer: $\boxed{14}$

   Solution: Since 2 minutes is 120 seconds, the instantaneous velocity is $f(120) = 120^2 + 5 = 14405$ meters per second. There are 1000 meters in a kilometer, so converting yields $14405 \div 1000 = \boxed{14.405}$ kilometers per second. After rounding to the nearest integer, we get $\boxed{14}$.

6. Find the largest prime divisor of $7^6 - 1$.

   Answer: $\boxed{43}$

   Solution: Factoring yields $7^6 - 1 = (7^3 - 1)(7^3 + 1) = (342)(344) = 171 \cdot 2 \cdot 43 \cdot 8 = 2^4 \cdot 3^2 \cdot 19 \cdot 43$. Thus, the answer is $\boxed{43}$.

7. Jack has 9 coins, consisting of only quarters, dimes, nickels, and pennies, with a total value of \$1.31. He has as many quarters as nickels and dimes combined, and he has half as many pennies as dimes. How many nickels does Jack have?

   Answer: $\boxed{2}$

   Solution: Since the dollar amount ends in a 1, we either have 1 penny or 6 pennies. It can't be 6 pennies, because it would be impossible to reach the remaining \$1.25 with three coins. So, we know that Jack has 1 penny, and thus he has 2 dimes. The remaining \$1.10 consists of 6 coins, some combination of nickels and quarters, and $n + d = q \implies n + 2 = q$, so there must be 4 quarters and $\boxed{2}$ nickels.

8. How many positive integer factors does 2520 have?

   Answer: $\boxed{48}$

   Solution: The prime factorization of 2520 is $2^3 \cdot 3^2 \cdot 5 \cdot 7$. To find the number of positive integer factors, add 1 to each power in the factorization and multiply. This yields $(3 + 1)(2 + 1)(1 + 1)(1 + 1) = \boxed{48}$ positive integer factors.

9. Convert 3120_4 to base 14. Do not include subscript base notation in your answer.

   Answer: $\boxed{116}$

   Solution: First, $3120_4 = (3 \cdot 4^3 + 1 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0)_{10} = 216_{10}$. We want to fit the highest power of 14 into 216. Since $14^2 = 196 < 216$, we have $216 = 1 \cdot 14^2 + 20$. The highest multiple of 14 that fits in 20 is 14, so $216 = 1 \cdot 14^2 + 1 \cdot 14^1 + 6 \cdot 14^0$. Thus, $3120_4 = 116_{14}$, and the answer is $\boxed{116}$.

10. One cold winter night, snow begins falling at a rate of 2 inches per hour, and Nathan begins shoveling his 20-foot by 20-foot driveway. He shovels and removes 25 square feet of snow every hour, regardless of the height of snow, and he always maximizes the volume of snow he removes. What is the volume of snow (in cubic feet) that is on Nathan's driveway after he shovels for 7 hours?

    Answer: $\boxed{350}$

    Solution: Divide the driveway into sixteen 5-foot by 5-foot squares. Every hour, Nathan will shovel one of these squares. He will only reach 7 of the 16 in 7 hours, so the other 9 squares have $7 \cdot 2 = 14$ inches of snow. At the 7-hour mark, the square he just finished shoveling has 0 inches of snow, the second to last has 2 inches, the third to last has 4 inches, and so on, and the first square he shoveled has 12 inches of snow. To find the volume, we multiply each of these heights by the area of the square, so making sure to adjust the units from inches to feet, we get $\frac{9 \cdot 14 + 2 + 4 + 6 + 8 + 10 + 12}{12} \cdot (5^2) = \boxed{350}$ cubic feet of snow.

11. If Bobby can eat a pizza in 30 minutes and Cassandra can eat a pizza in 15 minutes, how long, in minutes, will it take for Bobby and Cassandra to eat a pizza together?

    Answer: $\boxed{10}$

    Solution: Bobby would be able to eat 2 pizzas in an hour, and Cassandra would be able to eat 4 pizzas in an hour. Together, Bobby and Cassandra would eat 6 pizzas in an hour, so one pizza takes them $\boxed{10}$ minutes.

12. Jacob writes N, a three-digit multiple of 7, on a whiteboard, then covers one of the digits with a piece of paper. Julie knows that N is a multiple of 7. No matter which of the three digits are being covered, if Julie looks at the board, she can be certain of the value of the digit that is being covered. How many possible values of N are there?

Answer: $\boxed{12}$

Solution: The only way that Julie can be sure of the value of the covered digit is if there is only one possible value for that digit. In order for this to be true, the digit must be 3, 4, 5, or 6, since for any other value, it is possible to add or subtract 7 from the digit and get another three-digit multiple of 7 without changing the values of the other digits. The exception is the hundreds place, which can be 7, since 0 cannot be the leading digit.

The divisibility rule for 7 says that a three digit number $100a + 10b + c$ is divisible by 7 if and only if $10a + b - 2c$ is also divisible by 7. Since the values of a, b, and c are restricted, the minimum value of $10a + b - 2c$ is $10 \cdot 3 + 3 - 2 \cdot 6 = 21$ and the maximum is $10 \cdot 7 + 6 - 2 \cdot 3 = 70$. Thus, the possible values for $10a + b - 2c$ are 21, 28, 35, 42, 49, 56, 63, and 70. Also, $-9 \leq b - 2c \leq 0$ when b and c are 3, 4, 5, or 6, thus there is only one value of a for each value of $10a + b - 2c$. We work through the cases, looking for valid pairs of (b, c) for each value of $10a + b - 2c$ and a.

For $10a + b - 2c = 21$, $a = 3 \implies b - 2c = -9$, and we have $(b, c) = (3, 6)$.
For $10a + b - 2c = 28$, $a = 3 \implies b - 2c = -2$, and we have $(b, c) = (4, 3)$ and $(6, 4)$.
For $10a + b - 2c = 35$, $a = 4 \implies b - 2c = -5$, and we have $(b, c) = (3, 4)$ and $(5, 5)$.
For $10a + b - 2c = 42$, $a = 5 \implies b - 2c = -8$, and we have $(b, c) = (4, 6)$.
For $10a + b - 2c = 49$, $a = 5 \implies b - 2c = -1$, and we have $(b, c) = (5, 3)$.
For $10a + b - 2c = 56$, $a = 6 \implies b - 2c = -4$, and we have $(b, c) = (4, 4)$ and $(6, 5)$.
For $10a + b - 2c = 63$, $a = 7 \implies b - 2c = -7$, and we have $(b, c) = (3, 5)$ and $(5, 6)$.
For $10a + b - 2c = 70$, $a = 7 \implies b - 2c = 0$, and we have $(b, c) = (6, 3)$.

There are $\boxed{12}$ numbers for which this is true: 336, 343, 364, 434, 455, 546, 553, 644, 665, 735, 756, and 763.

13. Two cubes with integer side lengths have a combined volume of 7110. Find the sum of the possible sums of their side lengths.

Answer: $\boxed{30}$

Solution: First, notice that $7100 = 2 \cdot 3^2 \cdot 5 \cdot 79$. We want $a^3 + b^3 = 7110$. But, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, so we need to find the possible values for $a+b$ and figure out whether they have integer values.
Notice, however, that $a^2 - ab + b^2 = \frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2 \geq \frac{(a+b)^2}{4}$. Therefore, $(a+b)^3 \leq 28440$, which means that $a + b \leq 31$, as $31^3 = 961 \cdot 31 = 29791$. Furthermore, we notice that $0 < |a-b| < a+b$, since a, b are the same sign, so $(a+b)^3 \geq 7110$, and so thus that $a + b \geq 19$, as $18^3 = 324 \cdot 18 = 5832$. Therefore, we see that the possible values of $a+b$ are between 20 and 30. However, notice that, writing out the first several divisors of 7110, we have $2, 3, 5, 6, 9, 10, 15, 18, 30, 45$; but then we need $a+b = 30$, and $a^2 - ab + b^2 = 237$, which yields us with that $\frac{900}{4} + \frac{3}{4}(a-b)^2 = 237$, or that $\frac{3}{4}(a-b)^2 = 12$, meaning that $a - b = 4$, and so we have an integer solution.

Thus, our desired answer is $\boxed{30}$.

14. A police detective has an 80% chance of solving any crime. The detective is given 5 crimes to solve. The probability that the detective solves exactly 3 of the 5 crimes can be expressed as a ratio of two relatively prime positive integers, $\frac{a}{b}$. Find $a + b$.

Answer: $\boxed{753}$

Solution: The probability that the detective solves 3 out of 5 crimes is given by $\left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^2 = \frac{64}{3125}$. However, the detective may choose any 3 of the 5 crimes, which he can do in $\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$ ways. Thus, the final probability is $\frac{64}{3125} \cdot 10 = \frac{128}{625}$ and the answer is $128 + 625 = \boxed{753}$.

15. Roots r_1, r_2, and r_3 of the polynomial $x^3 - 5x^2 - 8x + a$ satisfy the equation $r_1 + 2r_2 + 4r_3 = 0$. What is the sum of the possible values of a?

Answer: $\boxed{60}$

Solution: Using Vieta's formulas, we obtain the following system of equations:

$$r_1 + r_2 + r_3 = 5 \tag{1}$$

$$r_1r_2 + r_2r_3 + r_3r_1 = -8 \tag{2}$$

$$r_1r_2r_3 = -a \tag{3}$$

$$r_1 + 2r_2 + 4r_3 = 0 \tag{4}$$

Then,

$$(4) - (1) \implies r_2 = -5 - 3r_3 \tag{5}$$

$$(4) - 2(1) \implies r_1 = 10 + 2r_3 \tag{6}$$

Substituting (5) and (6) into (2) gives

$$\begin{aligned}(10 + 2r_3)(-5 - 3r_3) + (-5 - 3r_3)r_3 + r_3(10 + 2r_3) &= -8,\\ r_3^2 + 5r_3 + 6 &= 0,\\ r_3 &= -2, -3.\end{aligned}$$

Substituting r_3 into (5), (6) and finally (3) yields $(r_1, r_2, r_3, a) = (6, 1, -2, 12), (4, 4, -3, 48)$. Thus, the answer is $12 + 48 = \boxed{60}$.