RAMC 2021

Middle School Team Solutions

Contest Problems/Solutions proposed by the Rochester Math Club problem writing committee:

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1. Pam has a cylindrical measuring cup with a diameter of 10 cm. The water level in the cup is currently at 4 cm. When Pam completely submerges three rocks of equal volume into the cup, the water level rises to 7 cm. If the volume of one of the rocks can be written as $a\pi$ cm^3, what is a?

   Answer: $\boxed{25}$

   Solution: The change in water level is $7 - 4 = 3$ cm, so the change in water volume is $3(\pi \cdot 5^2) = 75\pi$ cm^3. Thus, the volume of one rock is $75 \div 3 = \boxed{25}$.

2. Suppose that in a certain state, license plate numbers consist of four unique characters, two digits followed by two letters. George wants a license plate number where the two digits are in increasing numeric order, and the two letters on his plate are in alphabetical order (e.g. 07AC is valid, but 22GA is not). How many license plate numbers can George choose from?

   Answer: $\boxed{14625}$

   Solution: Since the order for both the digits and the letters are fixed, we just need to choose two of each that are distinct. There are $\binom{10}{2} = 45$ combinations of two distinct digits, and $\binom{26}{2} = 325$ combinations of two distinct letters. Thus, there are $45 \cdot 325 = \boxed{14625}$ plate numbers that George can choose from.

3. A box contains 17 blue balls, 15 red balls, 11 pink balls, 7 green balls, and 5 yellow balls. What is the least number of balls that must be drawn from the box without replacement to guarantee that at least 10 balls of the same color will be drawn?

   Answer: $\boxed{40}$

   Solution: First we draw as many balls as we can without exceeding 9 balls of the same color, which gives $9+9+9+7+5 = 39$ balls. On the next draw, there are only blue, red, and pink balls left, and then we can guarantee that we have drawn 10 balls of the same color. Thus, the answer is $39+1 = \boxed{40}$.

4. The 6-digit number $789ABC$ is divisible by 7, 8, and 9, and all of its digits are distinct. What is the 3-digit number ABC?

   Answer: $\boxed{264}$

   Solution: Being divisible by 7, 8, and 9 is the same as being divisible by $\text{lcm}(7,8,9)$, and since they are pair-wise coprime, $\text{lcm}(7,8,9) = 7 \cdot 8 \cdot 9 = 504$. The closest multiple of 504 that is at least 789000 is 789264, whose digits are all distinct. The next multiple is 789768, whose digits repeat. Thus, the answer is $\boxed{264}$.

5. Sally and Ella are playing a guessing game. Sally's number is n. She tells Ella that n and 50 have a greatest common factor of 5, and that n and 175 have a least common multiple of 1925. What is the sum of all possible values of n?

   Answer: $\boxed{440}$

   Solution: First, $\gcd(n, 50) = 5$ means that the power of 5 in the factorization of n must be 5^1, and also that n is odd. However, $\text{lcm}(n, 175) = 1925 = 5^2 \cdot 7 \cdot 11$ requires that n divides 1925, but since $175 = 5^2 \cdot 7$ isn't divisible by 11, this means n must be divisible by 11. Thus, n is either $5 \cdot 11 = 55$ or $5 \cdot 7 \cdot 11 = 385$, and the sum of these values is $55 + 385 = \boxed{440}$.

6. For a number N in base 10, the following holds true: $N_{10} = 763_b = 394_{b+5}$, where b is a positive integer. What is N?

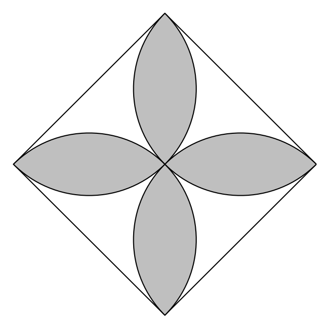
   Answer: $\boxed{916}$

   Solution: From the condition, we get

$$\begin{aligned} 7b^2 + 6b + 3 &= 3(b+5)^2 + 9(b+5) + 4, \\ 7b^2 + 6b + 3 &= 3b^2 + 30b + 75 + 9b + 45 + 4, \\ 4b^2 - 33b - 74 &= 0, \\ (b-11)(4b+11) &= 0. \end{aligned}$$

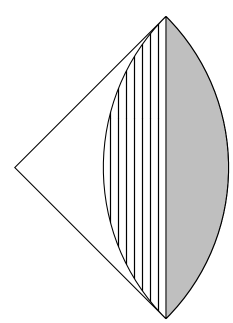
   Thus, $b = 11$ and $N = 7 \cdot 11^2 + 6 \cdot 11 + 3 = \boxed{916}$.

7. Four overlapping semi-circles are constructed inside a square of side length 12, with the sides as diameters, as shown. The area occupied by at least 2 semi-circles (shaded gray) can be expressed as $a\pi + b$. Find $a + b$.



   Answer: $\boxed{-72}$

   Solution: We begin by isolating a quarter-circle, shown on the right. Notice that half the area of this "petal" is given by the difference between the area of the quarter-circle and the area of the isosceles right triangle formed from its two radii. This area is $\frac{1}{4}\left(\pi \cdot 6^2\right) - \frac{1}{2} \cdot 6^2 = 9\pi - 18$. Since there are 8 "half-petals" we get $8\,(9\pi - 18) = 72\pi - 144$, so the answer is $72 - 144 = \boxed{-72}$.



8. The rule for divisibility by 109 is given by the following: take a positive integer, multiply the last digit by some positive integer k, then add it to the number formed by removing the last digit; if the result is divisible by 109, so is the original integer. Find the smallest value of k such that this is true.

Answer: $\boxed{11}$

Solution: We want $10a+b$ to be divisible by 109 if and only if $a+kb$ is divisible by 109. Notice that if $a+kb$ is divisible by 19, so is $10a+10kb$, and so is $(10a+10kb)-(10a+b)=b(10k-1)$. We can only choose the value of k, so the minimum value of k that guarantees that this is divisible by 109 is given by $10k-1=109 \implies k=\boxed{11}$.

9. The sum $\frac{1}{60}+\frac{1}{84}+\frac{1}{112}+\frac{1}{144}+\frac{1}{180}+\frac{1}{220}+\frac{1}{264}+\frac{1}{312}+\frac{1}{264}+\frac{1}{420}$ can be written as a ratio of two relatively prime positive integers, $\frac{a}{b}$. Find $a+b$.

Answer: $\boxed{42751}$

Solution: We have

$$\begin{aligned}
&\frac{1}{60}+\frac{1}{84}+\frac{1}{112}+\frac{1}{144}+\frac{1}{180}+\frac{1}{220}+\frac{1}{264}+\frac{1}{312}+\frac{1}{264}+\frac{1}{420},\\
&=\frac{1}{2}\left(\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{72}+\frac{1}{90}+\frac{1}{110}+\frac{1}{132}+\frac{1}{156}\right)+\frac{1}{264}+\frac{1}{420},\\
&=\frac{1}{2}\left(\frac{1}{5\cdot 6}+\frac{1}{6\cdot 7}+\cdots+\frac{1}{12\cdot 13}\right)+\frac{1}{264}+\frac{1}{420},\\
&=\frac{1}{2}\left(\left[\frac{1}{5}-\frac{1}{6}\right]+\left[\frac{1}{6}-\frac{1}{7}\right]+\cdots+\left[\frac{1}{12}-\frac{1}{13}\right]\right)+\frac{1}{264}+\frac{1}{420},\\
&=\frac{1}{2}\left(\frac{1}{5}-\frac{1}{13}\right)+\frac{1}{264}+\frac{1}{420},\\
&=\frac{4}{65}+\frac{1}{264}+\frac{1}{420},\\
&=\frac{2711}{40040}.
\end{aligned}$$

Thus, the answer is $2711+40040=\boxed{42751}$.

10. Cory's favorite number is a positive integer between 1 and 10000, inclusive. Lucas wants to figure out what this number is, and plays the following game: Lucas picks a number n between 1 and 100, inclusive, and asks Cory for the remainder when his favorite number is divided by n. Lucas repeats this until he knows for certain the value of Cory's favorite number. Let m be the least number of values Lucas must ask Cory about to figure out Cory's favorite number. Given that Lucas uses exactly m values, determine the smallest possible sum of the values of n that Lucas uses.

Answer: $\boxed{66}$

Solution: First, notice that $m > 2$ because the largest lcm possible is $\text{lcm}(100, 99) = 100 \cdot 99 = 9900 < 10000$, thus a remainder from 100 and 99 would not uniquely determine all the integers. So, $m \geq 3$, and, for example, $\text{lcm}(100, 99, 98) = 485100$, so we know that $m = 3$ is the minimum value.

To minimize the sum of the three numbers, we want numbers as close as possible. In particular, we require, by the AM-GM inequality, that $\left(\frac{a+b+c}{3}\right)^3 \geq 10000 \implies (a+b+c)^3 \geq 270000$. Since $64^3 = 262144$, we expect a smallest sum of 65. We can't have two even numbers, since then the lcm is at most half of the product of the three numbers, thus we need three odd numbers. The closest we can get is 19, 21, and 25, but $\text{lcm}(19, 21, 25) = 19 \cdot 21 \cdot 25 = 9975 < 10000$ so this doesn't work.

Thus we try a sum of 66, for which 21, 22, and 23 readily work, since $\text{lcm}(21, 22, 23) = 21 \cdot 22 \cdot 23 = 10626 > 10000$. So, the minimum sum is $\boxed{66}$.