Rochester Area Math Competition Solutions (Elementary School)

Hosted by Rochester Math Club (RMC)

March 14th, 2018

Not to be posted until **March 14th, 2018**, after 8:30pm CST.

1 Solutions

1. What is the area of a square with side length 9?

Solution: 81
The area of a square is s^2, where s is the side length. $9^2 = \mathbf{81}$.

2. In the magic square below, the numbers in each row and each column add up to the same sum. Find x.

Solution: 4
Adding up the numbers in the leftmost column, we attain a sum of 23. We can then solve for the number in the top right corner. Lastly, solve for x.

10.5	7	5.5
3.5		13.5
9		x= 4

3. A cake that costs 16.50 dollars is cut into 15 equal pieces. Tracy eats 3 pieces. How much are the pieces that Tracy ate worth?

Solution: \$3.30
Each piece of cake costs $16.50 \div 15 = \$1.10$. 3 pieces cost $3 \times 1.10 = \$\mathbf{3.30}$.

4. If 7 apples cost the same as 4 oranges, and 5 oranges cost the same as 2 grapefruits, how many apples cost the same as 16 grapefruits?

Solution: 70
Let's call the price of 1 apple a, the price of 1 orange o, and the price of 1 grapefruit g. With the given information, we have

$$7a = 4o$$

$$5o = 2g$$

To get the cost of 16 grapefruits, multiply the second equation by 8 to get $40o = 16g$. Notice $40o$ is 10 times $4o$, so multiply the first equation by 10 to get $70a = 40o$. By transitive property, $70a = 16g$, so **70** applies cost the same as 16 grapefruits.

5. A year on RMC Planet has exactly 420 days. Each week on this planet has 9 days. If year 1 starts on the first day of the week, what will be the next year that the first day of the year is the first day of the week?

Solution: Year 4
Since each week has 9 days, the number of days that passes before the next year with the first day of the week being the first day of the year must be a multiple of 9. Therefore, we are looking for the least common multiple of 9 and 420. Notice that 420 is a multiple of 3; multiplying it by 3 makes it a multiple of 9. That means 3 years must pass before the first day of the year is the first day of the week again, so this year is **year 4**.

6. Calculate the sum: $1 + 2 + 3 + \cdots + 48 + 49 + 50$.

Solution: 1275
Use the formula for the sum of arithmetic series: $\frac{(a_1+a_n)(n)}{2} = sum$, where a_1 and a_n are the first and last terms, respectively, and n is the number of terms. $\frac{(1+50)(50)}{2} = \mathbf{1275}$.

7. A bag has 7 red, 10 blue, and 8 white marbles. How many extra white marbles do you need to add so that the number of white marbles is half of the total number of marbles?

Solution: 9
There are $7 + 10 + 8 = 25$ marbles in the bag. Let x be the number of white marbles you need to add. Set up the equation $\frac{8+x}{25+x} = \frac{1}{2}$. Cross multiply, and you get $16 + 2x = 25 + x$, $x = \mathbf{9}$.

8. The special operation @ works as follows: $a@b = a^2 + b$. If $5@(3@x) = 49$, what is the value of x?

Solution: 15
Using the definition of @, rewrite $5@(3@x) = 49$ into $5@(3^2 + x) = 49$, then $5^2 + (3^2 + x) = 49$. Simplifying, we get $25 + 9 + x = 49$, $x = \mathbf{15}$.

9. Chris has 4 quarters (25 cents each), 7 dimes (10 cents each), 5 nickels (5 cents each), and 15 pennies (1 cent each). What is the greatest number of coins he can use to make $1.83?

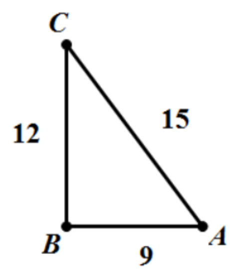
Solution: 28
The strategy here is to use as many low-value coins as possible. Note that $1.83 is equal to 183 cents. First, we should try to use all the pennies; that is 15 coins and 15 cents, leaving $183 - 15 = 168$ cents. Next, use all the nickels; that is $15 + 5 = 20$ coins in total and $15 + 5 \times 5 = 40$ cents, leaving 143 cents. Now, be careful, because quarters and dimes can only add up to values that are multiples

of 5, but 143 is not. We need to take away some coins that are already used. Taking away 2 pennies, we have used 18 coins and made up 38 cents, leaving 145 cents. Then, use all the dimes, making a total of $18 + 7 = 25$ coins leaving $145 - 7 \times 10 = 75$ cents. Use 3 quarters to make up the remaining 75 cents. The greatest number of coins Chris can use is $25 + 3 = \mathbf{28}$ coins.

10. The RMC orchard has trees arranged into a square with X columns and X rows. To enlarge the square by 2 rows and 2 columns, the RMC team planted 24 more trees. How many trees were there before the enlargement?

Solution: 25
To enlarge the square by 1 row and 1 column, the RMC team needs to plant $X + 1 + X + 1 - 1 = 2X + 1$ trees. We subtract 1 because the corner tree is counted twice. To enlarge it by another row and column, the RMC team needs to plant $(X + 1) + 1 + (X + 1) + 1 - 1 = 2X + 3$ trees. In total, the RMC team plants $2X + 1 + 2X + 3 = 24$ trees, $X = 5$. There were $5^2 = \mathbf{25}$ trees originally.

11. In the figure below, triangle ABC has side lengths $AB = 9, BC = 12$, and $CA = 15$. Find the length of the height of this triangle that passes through point B.



Solution: $\mathbf{\frac{36}{5}}$
Triangle ABC is a right triangle ($9^2 + 12^2 = 15^2$). The area of a triangle is $\frac{1}{2}bh$, so the area of triangle ABC is $\frac{1}{2}(9)(12) = 54$. To find the height that passes through point B, we need to use CA as the base. Call the height H, $\frac{1}{2}(15)H = 54$, $H = \mathbf{\frac{36}{5}}$.

12. Tracy, Richard, and Helen are each cutting a string. All three strings have the same integer length. Tracy makes 15 cuts, Richard makes 27 cuts, and Helen makes 9 cuts. Each initial string is cut into pieces of equal length. What is the smallest possible length of one of the original strings?

Solution: 560
First, notice that each cut results in one more piece of string than before. There-

fore, Tracy ends up with 16 pieces; Richard ends up with 28 pieces; Helen ends up with 10 pieces. The smallest possible length of one of the original strings is the least common multiple (LCM) of 16, 28, and 10. To find that, we will use the prime factorization of each number:

$$16 = 2^4$$
$$28 = 2^2 \times 7$$
$$10 = 2 \times 5$$

The LCM is $2^4 \times 5 \times 7 = \mathbf{560}$.

13. At a movie theater, a bag of popcorn costs twice as much as a bottle of soda. John bought 2 bags of popcorn and 4 bottles of soda, while Jane bought 4 bags of popcorn and 2 bottles of soda. John paid $2.50 less than Jane did. How much does a bag of popcorn cost?

Solution: $2.50
First, let's consider what John and Jane bought in common: 2 bags of popcorn and 2 bottles of soda. John bought 2 more bottles of soda, and Jane bought 2 more bags of popcorn, so the difference between 2 bottles of soda and 2 bags of popcorn is $2.50. We also know that a bag of popcorn costs twice as much as a bottle of soda, so 2 bottles of soda costs the same as 1 bag of popcorn. The difference between 1 bag of popcorn and 2 bags of popcorn is 1 bag of popcorn, and that is equivalent to $**2.50**.

14. Find exactly the infinite sum:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 15}+\cdots$$

Solution: $\frac{1}{2}$
Let's take a look at the first term in the sequence:

$$\frac{1}{1\cdot 3}=\frac{\frac{1}{1}-\frac{1}{3}}{2}$$

What about the second term?

$$\frac{1}{3\cdot 5}=\frac{\frac{1}{3}-\frac{1}{5}}{2}$$

In general, we see that $\frac{1}{n(2n+1)}=\frac{\frac{1}{n}-\frac{1}{2n+1}}{2}$. Thus, this infinite sum becomes:

$$\frac{\frac{1}{1}-\frac{1}{3}}{2}+\frac{\frac{1}{3}-\frac{1}{5}}{2}+\frac{\frac{1}{5}-\frac{1}{7}}{2}+\frac{\frac{1}{7}-\frac{1}{9}}{2}+\frac{\frac{1}{9}-\frac{1}{11}}{2}+\frac{\frac{1}{11}-\frac{1}{13}}{2}+\frac{\frac{1}{13}-\frac{1}{15}}{2}+\cdots$$

Taking out a $\frac{1}{2}$,

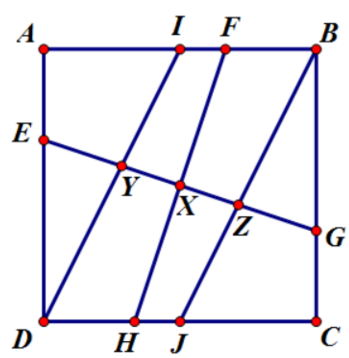
$$\frac{1}{2}(\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\frac{1}{7}-\frac{1}{9}+\frac{1}{9}-\frac{1}{11}\cdots)$$

This is called a *telescoping sequence*. Almost everything cancels. We see that everything but the first term, and eventually, the last term, are the only terms left.

$$=\frac{1}{2}(\frac{1}{1}-\frac{1}{\infty})=\frac{1}{2}(\frac{1}{1}-0)=\frac{1}{2}$$

15. In square $ABCD$ with side length 6, point E is on AD such that $AE:ED=1:2$, point F is on AB such that $BF:FA=1:2$, point G is on BC such that $CG:GB=1:2$, point H is on CD such that $DH:HC=1:2$. Lines EG and HF intersect at a point x. If I and J are the midpoints of AB and CD, respectively, and DI and BJ intersect line EG at Y and Z, respectively, find exactly $YX+XZ$.

Solution: $\mathbf{\frac{6\sqrt{10}}{7}}$



We use coordinate geometry. Assign point D as the origin. Our goal is to find the intersection of EG with the two lines DI and FH. From there, we find the distance between the two intersection points, and then double the distance for our final result. With $D(0,0)$, then $I(3,6)$, $F(4,6)$, $H(2,0)$, $E(0,4)$, and $G(6,2)$. Simple calculations lead us to find the equations of the respective lines:

Line EG: $-\frac{1}{3}x+4=y$
Line DI: $2x=y$ and HF: $3x-6=y$
Now, we just find the intersection of EG and DI.

$$2x=-\frac{1}{3}x+4$$

$$\frac{7}{3}x=4$$

$x = \frac{12}{7}$ and so $y = \frac{24}{7}$
Next: EG and HF.

$$3x - 6 = -\frac{1}{3}x + 4$$

$$\frac{10}{3}x = 10$$

$x = 3$ and $y = 3$ (actually, we could have known this without calculations; since both divide the sides into $1 : 2$ ratio, they will intersect in the middle of the square, which is $(3, 3)$. Thus, the distance between Y and Z is:

$$\sqrt{(3 - \frac{12}{7})^2 + (3 - \frac{24}{7})^2} = \sqrt{(\frac{9}{7})^2 + (-\frac{3}{7})^2}$$

$$= \sqrt{\frac{81}{49} + \frac{9}{49}} = \sqrt{\frac{90}{49}} = \frac{3\sqrt{10}}{7}$$

Doubling, our final solution is $\mathbf{\frac{6\sqrt{10}}{7}}$.