Rochester Area Math Competition Solutions (High School)

Hosted by Rochester Math Club (RMC)

March 14th, 2018

Not to be posted until **March 14th, 2018**, after 8:30pm CST.

1 Solutions

1. Given that $x = 3$, find exactly $3x^2 + \frac{x}{3} + 5 + \frac{2}{x}$.

Solution: $\mathbf{\frac{101}{3}}$

$$3(3)^2 + \frac{3}{3} + 5 + \frac{2}{3}$$
$$= 27 + 1 + 5 + \frac{2}{3}$$
$$= \mathbf{\frac{101}{3}}$$

2. Find all values of x that satisfy $2x^2 - 13x + 21 \leq 0$.

Solution: $\mathbf{3 \leq x \leq \frac{7}{2}}$ **or** $[\mathbf{3}, \mathbf{\frac{7}{2}}]$
First, consider the inequality as an equation. Factoring

$$(2x - 7)(x - 3) = 0$$
$$x = \frac{7}{2}, 3$$

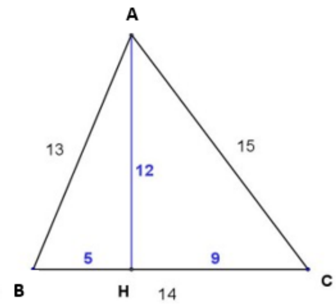
The graph of $2x^2 - 13x + 21 \leq 0$ is a parabola opening up, so the negative part (under the x-axis) is between the two zeros. Therefore, $\mathbf{3 < x < \frac{7}{2}}$ satisfies the inequality.

3. A sequence is called *special* if the next term in the sequence is the average of the two preceding terms (after the first two terms). For example, $100, 100, 100, 10...0$ and $100, 50, 75, 62.5...$ are both *special* sequences. Given that $6, 8, 7 \cdots$ is a *special* sequence, find exactly the 7th term of the sequence.

Solution: $\mathbf{\frac{117}{16}}$
Continuing with the rule, the 4th term is $\frac{7+8}{2} = \frac{15}{2}$, the 5th $\frac{\frac{15}{2}+7}{2} = \frac{29}{4}$, the 6th $\frac{\frac{29}{4}+\frac{15}{2}}{2} = \frac{59}{8}$, the 7th $\frac{\frac{59}{8}+\frac{29}{4}}{2} = \mathbf{\frac{117}{16}}$.

4. Triangle ΔABC has sides $AB = 13$, $BC = 14$, and $CA = 15$. A point H is on BC such that $AH \perp BC$. Find AH.

Solution: 12



Solution A: We use Heron's Formula to find the area of the triangle, and then from there, find the height. The semiperimeter is $\frac{13+14+15}{2} = 21$

$$Area = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(21-13)(21-14)(21-15)}$$

$$= \sqrt{21(8)(7)(6)} = \sqrt{7 \cdot 3 \cdot 2^3 \cdot 7 \cdot 3 \cdot 2} = \sqrt{7^2 \cdot 2^4 \cdot 3^2} = 7 \cdot 2^2 \cdot 3 = 84$$

So, since Area also is $\frac{1}{2}bh$, $84 = \frac{1}{2}(14)(AH)$, so $AH = \mathbf{12}$.

Solution B: Designate the length $AH = h$. Call $BH = x$, so $HC = 14 - x$. Now, we simply use the Pythagorean Theorem twice to determine h.

$$x^2 + h^2 = 13^2$$

$$(14-x)^2 + h^2 = 15^2$$

Guessing and checking leads us to find $h = 12$, or we can find it more systematically. Subtracting the first equation from the second,

$$-28x + 196 = 56$$

$$-28x = -140$$

$$x = 5$$

Plugging this back into the first equation,

$$5^2 + h^2 = 169$$

$$h^2 = 144$$

So $h = \mathbf{12}$.

5. If $\frac{5}{x+\frac{5}{x+\frac{5}{x+\cdots}}} = \frac{\sqrt{29}-3}{2}$, determine exactly the value of x.

Solution: 3
We see that the denominator looks familiar. In fact, disregarding the "x" in the denominator, the fractioned part is what the LHS is currently.
So,

$$\frac{5}{x+\frac{5}{x+\frac{5}{x+\cdots}}} \Rightarrow \frac{5}{x+\frac{\sqrt{29}-3}{2}}$$

$$\frac{5}{x+\frac{\sqrt{29}-3}{2}} = \frac{\sqrt{29}-3}{2}$$

$$5 = (x+\frac{\sqrt{29}-3}{2})(\frac{\sqrt{29}-3}{2})$$

$$5 = \frac{\sqrt{29}-3}{2}x + \frac{29-6\sqrt{29}+9}{4}$$

$$5 = \frac{\sqrt{29}-3}{2}x + \frac{19-3\sqrt{29}}{2}$$

$$10 = (\sqrt{29}-3)x + 19 - 3\sqrt{29}$$

$$3\sqrt{29} - 9 = 3(\sqrt{29}-3) = (\sqrt{29}-3)x$$

Finally, $x = \mathbf{3}$.

6. Towns A and B are 60 miles apart. Tracy and Richard start driving from A at the same time. They drive at constant speeds, but Tracy drives 10 mph faster than Richard does. Tracy reaches B after 1 hour and starts driving back to A. How far away from A does she pass Richard on the way back?

Solution: $\frac{\mathbf{600}}{\mathbf{11}}$ miles
If Tracy drives 60 miles in 1 hour, then her speed is 60 mph. Therefore, Richard's speed is 50 mph. When Tracy reaches Town B, Richard has driven 50 miles. Thus, when Tracy starts driving back immediately, the distance between them is 10 miles. Because they are now driving in opposite directions, the speed at which the distance between them is decreasing is the sum of their speeds, or $50 + 60 = 110$ mph. We can get that they meet $\frac{10miles}{110mph} = \frac{1}{11}hr$ after Tracy starts driving back. In that time, Richard drives $50mph \times \frac{1}{11}hr = \frac{50}{11}miles$. Adding to the previously driven 50 miles from Town A, we get $\frac{\mathbf{600}}{\mathbf{11}}$ miles.

7. If integers a and b, not necessarily distinct, are selected randomly and independently from $1-100$, find the probability that $a+b$ is even.

Solution: $\frac{\mathbf{1}}{\mathbf{2}}$
$a+b$ will be even under two conditions:

1. a and b are both even.
2. a and b are both odd.

The probability that a is even is $\frac{50}{100} = \frac{1}{2}$, and thus the probability that a is odd is $1 - \frac{1}{2} = \frac{1}{2}$. Since a and b are not necessarily distinct, it is the same probabilities for b.

The probability they are both even is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The probability they are both odd is similarly $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Thus, the probability that $a + b$ is even is simply $\frac{1}{4} + \frac{1}{4} = \mathbf{\frac{1}{2}}$

8. Given $a + b = 3$ and $a^2 + b^2 = 12$, find exactly the value of $a^6 + b^6$.

Solution: 1755
We square the first equation to find that $a^2 + 2ab + b^2 = 9$. Plugging in the second equation, $12 + 2ab = 9$, and $ab = -\dfrac{3}{2}$. We want to find $a^6 + b^6$. Both of these terms would appear if we multiplied out $(a^2 + b^2)(a^4 + b^4)$.

Expanding,

$$(a^2 + b^2)(a^4 + b^4) = a^6 + a^2b^4 + a^4b^2 + b^6 = a^6 + b^6 + (a^2b^2)(a^2 + b^2)$$

Thus, the only thing left to find is $a^4 + b^4$, which can simply be found my squaring $a^2 + b^2$.

$$a^2 + b^2 = 12$$

$$(a^2 + b^2)^2 = 12^2$$

$$a^4 + b^4 + 2ab = 144$$

We know $ab = -\dfrac{3}{2}$, and plugging this in,

$$a^4 + b^4 - 3 = 144$$

and

$$a^4 + b^4 = 147$$

So,

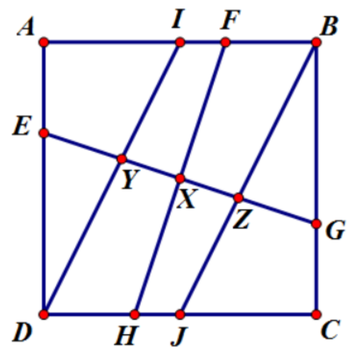
$$(a^2 + b^2)(a^4 + b^4) = 12 * 147 = a^6 + b^6 + ((-\frac{3}{2})^2)(12) = a^6 + b^6 + 9$$

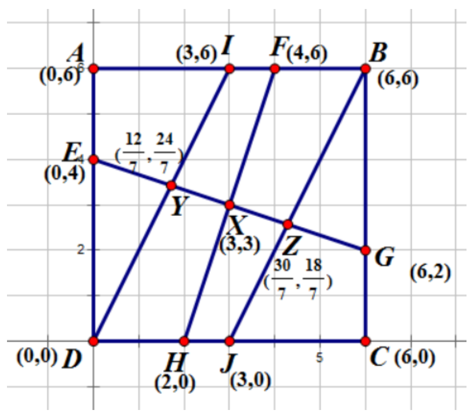
Doing some multiplication and subtracting both sides by 9, we reach our final answer of **1755**.

9. In square $ABCD$ with side length 6, point E is on AD such that $AE : ED = 1 : 2$, point F is on AB such that $BF : FA = 1 : 2$, point G is on BC such that $CG : GB = 1 : 2$, point H is on CD such that $DH : HC = 1 : 2$. Lines EG and HF intersect at a point x. If I and J are the midpoints of AB and

CD, respectively, and DI and BJ intersect line EG at Y and Z, respectively, find exactly $YX + XZ$.

Solution: $\mathbf{\frac{6\sqrt{10}}{7}}$





We use coordinate geometry. Assign point D as the origin. Our goal is to find the intersection of EG with the two lines DI and FH. From there, we find the distance between the two intersection points, and then double the distance for our final result. With $D(0,0)$, then $I(3,6)$, $F(4,6)$, $H(2,0)$, $E(0,4)$, and $G(6,2)$. Simple calculations lead us to find the equations of the respective lines:

Line EG: $-\frac{1}{3}x + 4 = y$
Line DI: $2x = y$ and HF: $3x - 6 = y$
Now, we just find the intersection of EG and DI.

$$2x = -\frac{1}{3}x + 4$$

$$\frac{7}{3}x = 4$$

$x = \frac{12}{7}$ and so $y = \frac{24}{7}$
Next: EG and HF.

$$3x - 6 = -\frac{1}{3}x + 4$$

$$\frac{10}{3}x = 10$$

$x = 3$ and $y = 3$ (actually, we could have known this without calculations; since both divide the sides into $1 : 2$ ratio, they will intersect in the middle of the square, which is $(3, 3)$. Thus, the distance between Y and Z is:

$$\sqrt{(3 - \frac{12}{7})^2 + (3 - \frac{24}{7})^2} = \sqrt{(\frac{9}{7})^2 + (-\frac{3}{7})^2}$$

$$= \sqrt{\frac{81}{49} + \frac{9}{49}} = \sqrt{\frac{90}{49}} = \frac{3\sqrt{10}}{7}$$

Doubling, our final solution is $\boldsymbol{\frac{6\sqrt{10}}{7}}$.

10. How many five-digit positive integers have digits that strictly increase or strictly decrease? For example, 12345 is a strictly increasing integer, but 12334 is not.

Solution: 378
Let's first look at strictly increasing numbers. We see that given any 5 distinct digits, there is only one way to have it such that the digits are strictly increasing (for example, the digits $3, 4, 6, 2, 5$ can only be arranged in a strictly increasing fashion in one way: 23456). We only have 9 distinct digits for strictly increasing integers, as 0 is not a viable digit. Thus, for strictly increasing integers, we just choose 5 digits from 9, which is simply $\binom{9}{5}$.

Using the same logic, strictly decreasing is choosing 5 digits but from a total of 10, as 0 can now be used, or $\binom{10}{5}$. Our answer is then $\binom{9}{5} + \binom{10}{5} = 378$.

11. Six distinguishable people are sitting around a circular table, each holding a fair coin. All six people flip their coins and those who flip tails stand while those who flip heads remain seated. What is the probability that no two adjacent people will stand?

Solution: $\boldsymbol{\frac{9}{32}}$
We do casework. We see that the maximum number of tails is 3, so we list out the cases of 0, 1, 2 and 3 tails.

Case 1: 0 people are standing. This gives us just 1 arrangement.

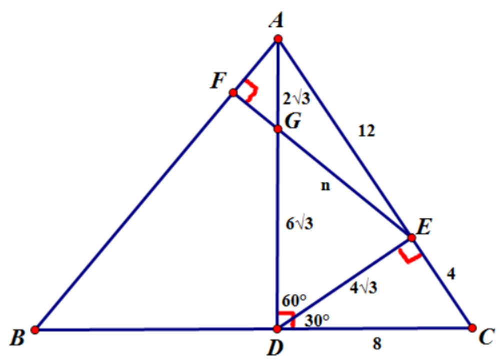
Case 2: 1 person is standing. This gives us 6 arrangements.

Case 3: 3 people are standing. The arrangement must be sit-stand-sit-stand-sit-stand or vice-versa, giving us just 2 arrangements.

Case 4: 2 people are standing. We see that we can choose any 2 people from the 6, and then subtract the cases where they don't work. The cases that don't work is when there are two people right next to each other, which occurs 6 times. Thus, this is $\binom{6}{2} - 6 = 9$ arrangements.
Summing, we get $1 + 6 + 2 + 9 = 18$ total. There are 2^6 arrangements, so our answer is $\frac{18}{64} = \mathbf{\frac{9}{32}}$

13. In triangle ABC, points D, E, F are on BC, CA, AB, respectively, such that $AD \perp BC$, $DE \perp CA$, $EF \perp AB$. AD and EF intersect at G. Given that $CE = 4$, $EA = 12$, and that $AG = 2\sqrt{3}$, find EG.

Solution: $\mathbf{2\sqrt{21}}$



Refer to the figure. We see that since $CE = 4$ and $EA = 12$, $DE = \sqrt{4 \cdot 12} = \sqrt{48} = 4\sqrt{3}$. Now, since $CE = 4$, we notice that ΔDEC is a $30-60-90$ triangle, and $\angle D = 30°$. Thus, since $AD \perp DC$, then $\angle ADE = 60°$. Further, since $AC = 16$ and $\angle C = 60°$, $AD = 8\sqrt{3}$. We are given that $AG = 2\sqrt{3}$, so $GD = 6\sqrt{3}$.
In the end, we want to find EG, which can now be found using law of cosines on triangle GED.

$$GE^2 = GD^2 + ED^2 - 2(GD)(ED)\cos(GDE)$$

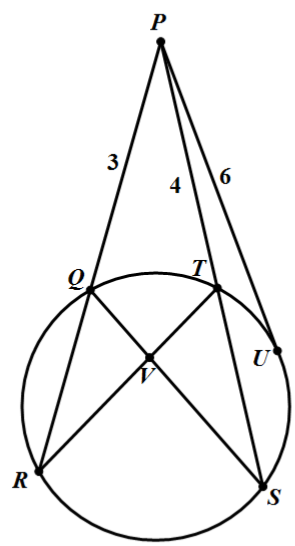
$$GE^2 = (6\sqrt{3})^2 + (4\sqrt{3})^2 - 2(6\sqrt{3})(4\sqrt{3})\cos(60°)$$

$$GE^2 = 108 + 48 - 2(72)(\frac{1}{2}) = 156 - 72 = 84$$

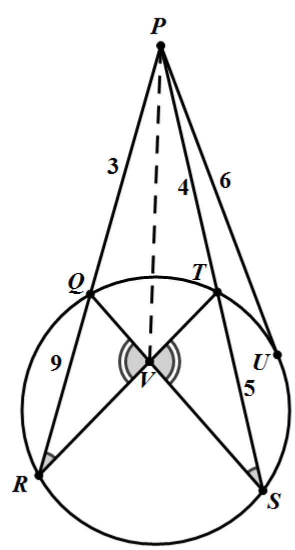
Finally,

$$GE = \sqrt{84} = \mathbf{2\sqrt{21}}$$

13. In the figure below, points Q, R, S, and T lie on the circle. If PU is tangent to the circle and has length 6, $PQ = 3$ and $PT = 4$, determine exactly the ratio of the area of quadrilateral $PQVT$ to the area of ΔQRV.



Solution: $\mathbf{\frac{47}{81}}$



Since PU is tangent, we can use Power of a point to find QR and TS.

$$6^2 = 3(3 + QR)$$

$$6^2 = 4(4 + TS)$$

So $QR = 9$ and $TS = 5$.

Our goal is to find the area of quadrilateral $PQVT$, which we can divide into two triangles. We see that triangles PQV and QVR actually have the same height (as their bases lie on the same lie), so we see that the ratio of the areas between the two triangles is the ratio of the bases, or $3 : 9 = 1 : 3$. Further, we see that triangles QRV and TVS are similar. Then, by side-area relationships, the area of TVS is $(\frac{5}{9})^2 = \frac{25}{81}$ of the area of QVR. We see that triangles PTV and TVS actually have the same height (as their bases lie on the same lie), so we see that the ratio of the areas between the two triangles is the ratio of the bases, or $4 : 5$. That means that $[PTV] = \frac{4}{5} \cdot \frac{25}{81} = \frac{20}{81}$ of the area of QVR.

Thus, with respect to the area of QVR, the area of the quadrilateral $PQVT = [PQV] + [PTV] = (\frac{1}{3} + \frac{20}{81})[QVR] = \frac{47}{81}[QVR]$. The ratio is simply $\mathbf{\frac{47}{81}}$.

14. The roots of the polynomial $x^3 + 4x - 1 = 0$ are r_1, r_2 and r_3. Find exactly

$$\frac{2r_1^2}{(3r_2+1)(3r_3+1)} + \frac{2r_2^2}{(3r_1+1)(3r_3+1)} + \frac{2r_3^2}{(3r_1+1)(3r_2+1)}$$

.

Solution: $\mathbf{\frac{1}{32}}$

Let's just take out the 2 from the expression and multiply it back at the end. Let's also combine denominators.

$$\frac{r_1^2}{(3r_2+1)(3r_3+1)} \cdot \frac{3r_1+1}{3r_1+1} + \frac{r_2^2}{(3r_1+1)(3r_3+1)} \cdot \frac{3r_2+1}{3r_2+1} + \frac{r_3^2}{(3r_1+1)(3r_2+1)} \cdot \frac{3r_3+1}{3r_3+1}$$

So

$$= \frac{3r_1^3 + r_1^2}{(3r_1+1)(3r_2+1)(3r_3+1)} + \frac{3r_2^3 + r_2^2}{(3r_1+1)(3r_2+1)(3r_3+1)} + \frac{3r_3^3 + r_3^2}{(3r_1+1)(3r_2+1)(3r_3+1)}$$

Now we can combine. Grouping,

$$= \frac{3(r_1^3 + r_2^3 + r_3^3) + (r_1^2 + r_2^2 + r_3^2)}{(3r_1+1)(3r_2+1)(3r_3+1)}$$

Looking at this expression, we see that we want to find the sum of the cubes of the roots and also the sum of the squares of the roots. This can be done through Newton Sums and Vieta's, which we get from our equation.

Going back$\Rightarrow$ $x^3 + 4x - 1 = 0$ tells us that (with Vieta's):

$$r_1 + r_2 + r_3 = 0 \tag{1}$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = 4 \tag{2}$$

$$r_1 r_2 r_3 = 1 \tag{3}$$

We first find $r_1^2 + r_2^2 + r_3^2$, which can be done by squaring (1).

$$(r_1 + r_2 + r_3)^2 = r_1^2 + r_2^2 + r_3^2 + 2(r_1 r_2 + r_1 r_3 + r_2 r_3)$$

So

$$0^2 = r_1^2 + r_2^2 + r_3^2 + 2(4)$$

and

$$-8 = r_1^2 + r_2^2 + r_3^2$$

Now we move onto $r_1^3 + r_2^3 + r_3^3$. Instead of cubing the first term, which would be overly-strenuous (and really not fun), we notice that since $r_1 + r_2 + r_3 = 0$, then $r_3 = -r_1 - r_2$.

We can use this fact and substitute it into what we want to find.

$$r_1^3 + r_2^3 + r_3^3 = r_1^3 + r_2^3 + (-r_1 - r_2)^3$$

Expanding the parenthesized value using the binomial theorem (or just hand expansion) finds:

$$r_1^3 + r_2^3 + (-r_1 - r_2)^3 = r_1^3 + r_2^3 - (r_1^3 + 3r_1^2 r_2 + 3r_1 r_2^2 + r_2^3)$$

Multiplying the negative in, some things will cancel; we are left with a simpler expression.

$$r_1^3 + r_2^3 - (r_1^3 + 3r_1^2 r_2 + 3r_1 r_2^2 + r_2^3)$$
$$= r_1^3 + r_2^3 - r_1^3 - 3r_1^2 r_2 - 3r_1 r_2^2 - r_2^3 = -3r_1^2 r_2 - 3r_1 r_2^2$$
$$= -3r_1 r_2 (r_1 + r_2)$$

But again using (1), $r_1 + r_2 = -r_3$, so

$$= -3r_1 r_2 (r_1 + r_2) = -3r_1 r_2 (-r_3) = 3(r_1 r_2 r_3)$$

Using (3), this value is simply 3, so $r_1^3 + r_2^3 + r_3^3 = 3$ We can finally plug these values back in.

$$\frac{3(r_1^3 + r_2^3 + r_3^3) + (r_1^2 + r_2^2 + r_3^2)}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)} = \frac{3(3) + (-8)}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)}$$

$$= \frac{1}{(3r_1 + 1)(3r_2 + 1)(3r_3 + 1)}$$

The only object left to tackle is the denominator. Simple expansion reveals that $(3r_1+1)(3r_2+1)(3r_3+1) = 27(r_1r_2r_3)+9(r_1r_2+r_1r_3+r_2r_3)+1 = 27(1)+9(4)+1 = 64$ So,

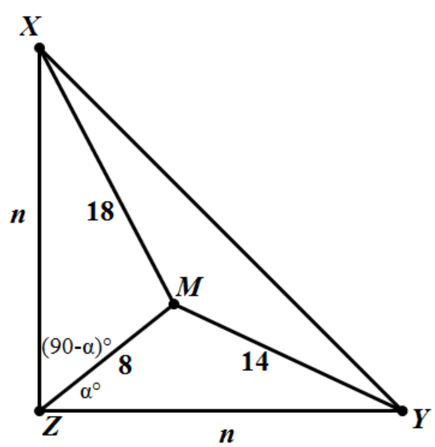
$$\frac{1}{(3r_1+1)(3r_2+1)(3r_3+1)} = \frac{1}{64}$$

But let's not forget about the 2 we factored out at the beginning! Our answer is $2 \cdot \frac{1}{64} = \mathbf{\frac{1}{32}}$.

15. Isosceles $\triangle XYZ$ has a right angle at Z. Point M is inside $\triangle XYZ$, such that $MX = 18$, $MY = 14$, and $MZ = 8$. If legs $\overline{XZ}$ and $\overline{YZ}$ have length n, find n exactly.

Solution: $\mathbf{\sqrt{260+112\sqrt{2}}}$

Solution A:



Using the figure above, we perform law of cosines on the angle Z.

$$18^2 = n^2 + 8^2 - 2(n)(8)\cos(90-\alpha)$$

But $\cos(90-\alpha) = \sin(\alpha)$ so $18^2 = n^2 + 8^2 - 2(n)(8)\sin(\alpha)$. The other equation is:

$$14^2 = n^2 + 8^2 - 2(n)(8)\cos(\alpha)$$

We need to use the fact that $\cos^2(\alpha) + \sin^2(\alpha) = 1$, so we will have to single out the trig parts.

$$2(n)(8)\sin(\alpha) = n^2 + 8^2 - 18^2$$

$$\sin(\alpha) = \frac{n^2 + 8^2 - 18^2}{2(n)(8)} = \frac{n^2 - 260}{16n}$$

Similarly,

$$\cos(\alpha) = \frac{n^2 + 8^2 - 14^2}{2(n)(8)} = \frac{n^2 - 132}{16n}$$

Squaring the two equations and adding them together,

$$1 = (\frac{n^2 - 260}{16n})^2 + (\frac{n^2 - 132}{16n})^2$$

Solving,

$$1 = \frac{n^4 - 520n^2 + 260^2}{256n^2} + \frac{n^4 - 264n^2 + 132^2}{256n^2}$$

$$256n^2 = 2n^4 - 784n^2 + 80524$$

$$128n^2 = n^4 - 392n^2 + 42512$$

$$n^4 - 520n^2 + 42512 = 0$$

Use the quadratic formula to solve for n^2:

$$n^2 = \frac{520 \pm \sqrt{(-520)^2 - 4(42512)}}{2}$$

$$n^2 = \frac{520 \pm \sqrt{100352}}{2}$$

$$n^2 = \frac{520 \pm 224\sqrt{2}}{2}$$

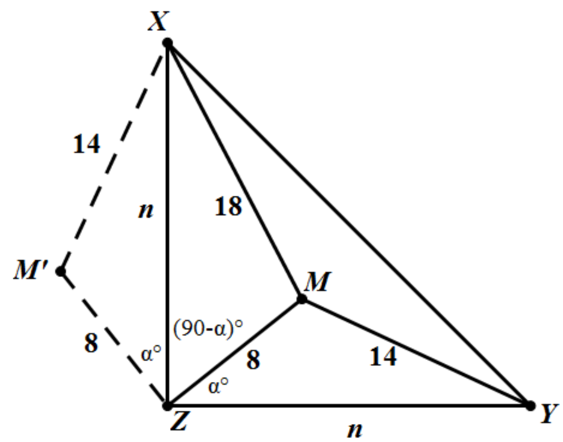
$$n^2 = 260 \pm 112\sqrt{2}$$

$$n = \pm\sqrt{260 \pm 112\sqrt{2}}$$

The negative solution is extraneous, so $n = \sqrt{260 \pm 112\sqrt{2}}$, which can be simplified down to $\mathbf{2\sqrt{65 + 28\sqrt{2}}}$. Both are acceptable.

Note: $n = \sqrt{260 - 112\sqrt{2}}$ makes it such that 18 is the largest side, which would contradict with the fact that $\angle XMZ, YMZ, YMX$ are obtuse; we prove this is in **Solution B**.

Solution B:



Rotate triangle ZYM counterclockwise by $90°$ such that Y is now Z (refer to the figure). Since the triangle is isosceles, we have the the base of the triangles align. Further, since $M'Z = MZ = 8$ and $M'ZM = 90°$, $M'M = 8\sqrt{2}$. Now, we test triangle $M'XM$. In fact, we see that $(8\sqrt{2})^2 + 14^2 = 18^2$, so $XM'M = 90°$. Thus, $\angle XM'Z = 45° + 90° = 135°$. Now, we can perform simple law of cosines on the triangle $XM'Z$ to find the length of the side of the triangle.

$$n^2 = 8^2 + 14^2 - 2(8)(14)\cos 135$$

$$n^2 = 64 + 196 - 2(8)(14)(-\frac{\sqrt{2}}{2})$$

$$n^2 = 260 + 112\sqrt{2}$$

So $n = \sqrt{\mathbf{260 + 112\sqrt{2}}}$. This is reasonably simplified.