Rochester Math Circle Placement Test (Advanced) Solutions

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1 Solutions

1. Calculate exactly the sum $1 + 3 + 5 + 7 + ... + 37 + 39$.

Solution: 400

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$

We notice that the sum of the first n odd numbers equals n^2. Since there are 20 terms in this summation, then the total sum is 20^2 or **400**.
Alternative solution: use the arithmetic sum formula.

$$\frac{(1 + 39)(20)}{2} = \mathbf{400}$$

2. Find the 32nd term of the arithmetic sequence $13, 24, 35, 46...$

Solution: 354
We observe that the common difference of this arithmetic sequence is $d = 11$. Using the explicit arithmetic sequence formula, $a_n = a_1 + (n-1)d$, and plugging in the numbers $a_{32} = 13 + (32 - 1)(11)$, we receive the result $a_{32} = \mathbf{354}$.

3. Equilateral triangle ABE is constructed inside unit square $ABCD$. Find exactly the area inside the square but outside the triangle.

Solution: $\mathbf{1 - \frac{\sqrt{3}}{4}}$
The area of the square is $1^2 = 1$, and the area of an equilateral triangle with length 1 is $\frac{\sqrt{3}}{4}$. Thus, the area we want is simply $\mathbf{1 - \frac{\sqrt{3}}{4}}$.

4. Daniel flips four fair coins. If at least one lands heads, he wins the lottery.

What is the probability that he wins?

Solution: $\mathbf{\frac{15}{16}}$
We will use complementary counting for this problem. The only case in which he doesn't win the lottery is if he flips all tails, which has a probability of $(\frac{1}{2})^4 = \frac{1}{16}$. We then subtract that from 1 to get the answer of $\mathbf{\frac{15}{16}}$.

5. Find exactly all values of x in the equation $\sqrt{x^2+8}+x=6$

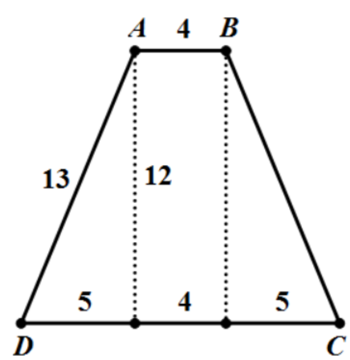
Solution: $\mathbf{\frac{7}{3}}$

$$\begin{aligned}\sqrt{x^2+8} &= 6-x\\ x^2+8 &= 36-12x+x^2\\ 8 &= 36-12x\\ 12x &= 28\\ x &= \frac{28}{12} = \mathbf{\frac{7}{3}}\end{aligned}$$

Let's check if this solution is extraneous. Plug $\frac{7}{3}$ back into the original equation, we get $\sqrt{(\frac{7}{3})^2+8}+\frac{7}{3}=6$. $\sqrt{\frac{121}{9}}+\frac{7}{3}=\frac{11}{3}+\frac{7}{3}=6$. This solution is valid.

6. Trapezoid $ABCD$ has side lengths $AB=4$, $CD=14$, and $BC=AD=13$. Find exactly the area of $ABCD$.

Solution: 108



Drop heights from points A and B. CD is divided into three segments of lengths 5, 4, and 5. The height is 12, which can be found through Pythagorean Theorem or knowledge of the Pythagorean triple $(5,12,13)$.The area of $ABCD$ is $(4+14)(12)(\frac{1}{2})=\mathbf{108}$.

7. 630 tokens are arranged into a triangular shape such that the first row has 1 token, the second row has 2 tokens, the third row has 3 tokens, and so on.

How many tokens are there in the last row?

Solution: 35
We can look at this as an arithmetic series with $a_1 = 1$, $d = 1$, and a sum of 630. Set up an equation with n representing the last term:

$$\begin{aligned}
\frac{n(n+1)}{2} &= 630 \\
n(n+1) &= 1260 \\
n^2 + n - 1260 &= 0 \\
1260 &= 5 \times 6^2 \times 7 = 35 \times 36 \\
(n+36)(n-35) &= 0 \\
n &= 35
\end{aligned}$$

There are **35** tokens in the last row.

8. Suppose that x and y are numbers such that $\sin(x+y) = 0.7$ and $\sin(x-y) = 0.3$. Find exactly $\sin x \cdot \cos y$.

Solution: 0.5
$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$ and $\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x)$ By adding these two identities together, we get $2\sin(x)\cos(y)$. However, we are given that $\sin(x+y) = 0.7$ and $\sin(x-y) = 0.3$, so $2\sin(x)\cos(y) = 1$ and $\sin(x)\cos(y) = 0.5$.

9. There are 4 red cards, 3 blue cards, and 2 white cards. All cards of the same color are identical. Five cards are chosen to be arranged in a linear fashion. How many different arrangements are possible?

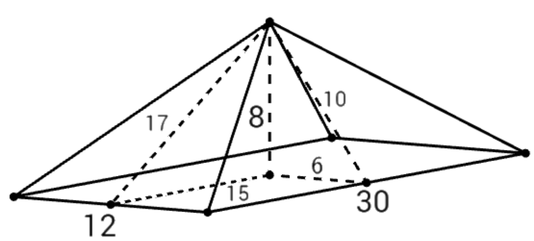
Solution: 180
Start by listing out all the possible color combinations, then permute each case. You can think of each color as a letter

Red	Blue	White	Permutations
4	1	0	$\frac{5!}{4!} = 5$
4	0	1	$\frac{5!}{4!} = 5$
3	2	0	$\frac{5!}{3!2!} = 10$
3	1	1	$\frac{5!}{3!} = 20$
3	0	2	$\frac{5!}{3!2!} = 10$
2	3	0	$\frac{5!}{2!3!} = 10$
2	2	1	$\frac{5!}{2!2!} = 30$
2	1	2	$\frac{5!}{2!2!} = 30$
1	3	1	$\frac{5!}{3!} = 20$
1	2	2	$\frac{5!}{2!2!} = 30$
0	3	2	$\frac{5!}{3!2!} = 10$

Add all the permutations together to receive the answer of **180**.

10. A pyramid has a rectangular base with dimensions 12 by 30 and a height of 8. Calculate its surface area.

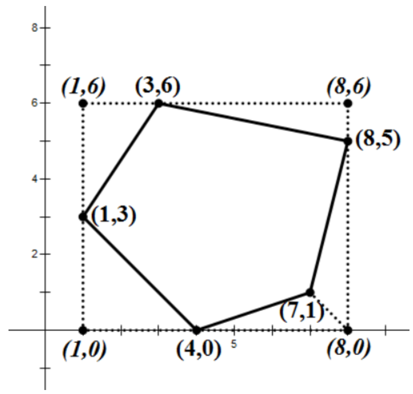
Solution: 864



Connect the point at which the heights hits the rectangular base to the midpoints of the width and length of the base. Then, connect the vertex to the midpoints of the width and length. Two right triangles are created, and we can use Pythagorean Theorem to calculate the slant heights. $\sqrt{8^2 + 6^2} = 10$. $\sqrt{8^2 + 15^2} = 17$. Knowledge of the Pythagorean triples $(6, 8, 10)$, or $(3, 4, 5)$, and $(8, 15, 17)$ would make this process faster. Proceed to calculate the surface area. $(12)(30) + (2)(\frac{1}{2})(12)(17) + (2)(\frac{1}{2})(30)(10) = \mathbf{864}$.

11. A pentagon in the coordinate plane has vertices at $(1,3)$, $(3,6)$, $(8,5)$, $(7,1)$, and $(4,0)$. Calculate its area.

Solution: 27.5 or $\frac{55}{2}$



Method one: inscribe the pentagon in a rectangle and subtract the areas of the extra triangles. The area of the rectangle is $6*7=42$. All of the triangles' areas can be found through $(\frac{1}{2})(base)(height)$. $42-(\frac{1}{2})(3)(3)-(\frac{1}{2})(2)(3)-(\frac{1}{2})(5)(1)-(\frac{1}{2})(5)(1)-(\frac{1}{2})(4)(1)=\mathbf{27.5}$.
Method two: use shoelace. $\frac{1}{2}|[(1)(6)+(3)(5)+(8)(1)+(4)(3)]-[(3)(3)+(6)(8)+(5)(7)+(1)(4)+(0)(1)]|=\mathbf{27.5}$. For more information on the shoelace technique, see `http://artofproblemsolving.com/wiki/index.php?title=Shoelace_Theorem`

12. In how many ways is it possible to seat seven people (Alex, Bill, Caroline, Dillon, Elie, Frank, Gertrude) at a round table if Alex and Bill must not sit in adjacent seats?

Solution: 480
There are $6!=720$ total ways to arrange the seven people, if we ignore the condition.
Now, we follow through on the condition. Assume that Alex and Bill are actually just one person. Then, we just need to arrange 6 people around a table, or $5!=120$. However, Alex and Bill can sit in two ways, so there are 240 ways to arrange them around a table. All these cases are invalid. Thus, the number of valid cases equals $720-240=480$.

13. Find all possible ordered pairs (x,y), where x and y are positive integers, that satisfy $2xy+6x-5y=36$.

Solution: (3,18), (4,4)
Using Simon's Favorite Factoring Trick, the given equation can be factored into

$(2x-5)(y+3)=21$. 21 has 2 pairs of factors, $(1,21)$, $(3,7)$ Set up equations for the first pair of factors:

$$2x-5=1, y+3=21$$
$$x=3, y=18$$
$$\mathbf{(3,18)}$$

Notice that $2x-5=21$ and $y+3=1$ will not yield results that satisfy the problem's conditions. Set up equations for the second pair of factors:

$$2x-5=3, y+3=7$$
$$x=4, y=4$$
$$\mathbf{(4,4)}$$

Notice again that $2x-5=7$ and $y+3=3$ will not satisfy the problem's conditions.
Alternative solution: solve for x after factoring.

$$2x-5=\frac{21}{y+3}$$
$$x=\frac{1}{2}(\frac{21}{y+3}+5)$$

Since x and y are positive integers, $\frac{21}{y+3}$ must be an integer as well.$y+3$ equals to a factor of 21, which include $1,3,7,21$. For y to be positive, it can only be 4 or 18. Plug those values in to attain the x values of 4 and 3. $(\mathbf{4},\mathbf{4}),(\mathbf{3},\mathbf{18})$

14. Calculate exactly the sum $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+...+\frac{1}{\sqrt{899}+\sqrt{900}}$

Solution: 29
This problem requires the telescoping technique.Start by rationalizing the first few fractions:

$$(\frac{1}{\sqrt{1}+\sqrt{2}})(\frac{\sqrt{1}-\sqrt{2}}{\sqrt{1}-\sqrt{2}})=\frac{\sqrt{1}-\sqrt{2}}{-1}$$
$$(\frac{1}{\sqrt{2}+\sqrt{3}})(\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}})=\frac{\sqrt{2}-\sqrt{3}}{-1}$$

Notice that if you add the two fractions together, the $-\sqrt{2}$ and $\sqrt{2}$ cancel out. Continuing this process, all terms cancel out except for the first and last terms.

$$\frac{\sqrt{1}\cancel{-\sqrt{2}}}{-1}+\frac{\cancel{\sqrt{2}}\cancel{-\sqrt{3}}}{-1}+\frac{\cancel{\sqrt{3}}-\sqrt{4}}{-1}+...+\frac{\sqrt{898}\cancel{-\sqrt{899}}}{-1}+\frac{\cancel{\sqrt{899}}-\sqrt{900}}{-1}$$
$$\frac{\sqrt{1}-\sqrt{900}}{-1}=\frac{1-30}{-1}=\mathbf{29}$$

15. For a positive integer n, let $d(n)$ denote the number of divisors of n. Find all ordered pairs of primes (p, q) for which

$$p \cdot d(pq) + q \cdot d(13pq) = 76.$$

Solution: $(5, 7)$
The number of divisors of a number is the product of the prime factor's exponents of the number plus one. In other words, a prime number, say 3, has $3^{1+1} = 1 + 1 = 2$ factors (namely 1 and 3). For example then, $12 = 2^2 \cdot 3$. So the number of factors of this number is equal to: $2^{2+1} \cdot 3^{1+1} = (2 + 1)(1 + 1) = 3 * 2 = 6$, namely $1, 2, 3, 4, 6, 12$.
Then, for this question, we have to test four conditions:
Case 1: $p \neq q \neq 13$

$$d(pq) = 2 * 2 = 4$$

$$d(13pq) = 2 * 2 * 2 = 8$$

$$4p + 8q = 76$$

$$p + 2q = 19$$

Since $2q$ is always even, we will have to test each prime between 3 and 19.
When $p = 3$, $3 + 2q = 19$ and $q = 8$ Nope.
When $p = 5$, $5 + 2q = 19$ and $q = 7$ Yep.
When $p = 7$, $7 + 2q = 19$ and $q = 6$ Nope.
When $p = 11$, $11 + 2q = 19$ and $q = 4$ Nope.
When $p = 13$, $13 + 2q = 19$ and $q = 3$ Nope–if $p = 13$, this breaks our initial condition that $p \neq 13$.
When $p = 17$, $17 + 2q = 19$ and $q = 1$ Nope.
When $p = 19$, $19 + 2q = 19$ and $q = 0$ Nope.
Case 2: $p = q \neq 13$

$$d(pq) = d(p^2) = 3$$

$$d(13pq) = d(13p^2) = 6$$

$$3p + 6q = 76$$

76, however, is not divisible by 3, so no cases work here.

Case 3: $p = 13$, $q \neq 13$

$$d(pq) = 4$$

$$d(13pq) = d(13^2 q) = 3 \cdot 2 = 6$$

$$4p + 6q = 76$$

$$52 + 6q = 76$$
$$q = 4$$

Nope.
Case 4: $q = 13$, $p \neq 13$

$$d(pq) = d(p^2) = 3$$
$$d(13pq) = d(13p^2) = 6$$
$$3p + 6q = 76$$

76, however, is not divisible by 3, so no cases work here. **Case 3:** $p = 13$, $q \neq 13$

$$d(pq) = 4$$
$$d(13pq) = d(13^2p) = 3 \cdot 2 = 6$$
$$4p + 6q = 76$$
$$4p + 78 = 76$$
$$q = -\frac{1}{2}$$

Nope.
Thus, the only condition that works is $(5, 7)$, and is our final answer.

16. $f(x) = \frac{x^2+x}{x^4-1} + \frac{1}{x^4-x^3+x^2-x}$, calculate exactly the value $f(2) + f(3) + f(4) + \ldots + f(2017)$.

Solution: $\mathbf{\frac{2016}{2017}}$
Start by simplifying $f(x)$.

$$\begin{aligned}
f(x) &= \frac{x(x+1)}{(x^2+1)(x+1)(x-1)} + \frac{1}{x(x^3-x^2+x-1)} \\
&= \frac{x}{(x^2+1)(x-1)} + \frac{1}{x(x^2(x-1)+(x-1))} \\
&= \frac{x}{(x^2+1)(x-1)} + \frac{1}{x(x^2+1)(x-1)} \\
&= \frac{x^2+1}{x(x^2+1)(x-1)} \\
f(x) &= \frac{1}{x(x-1)}
\end{aligned}$$

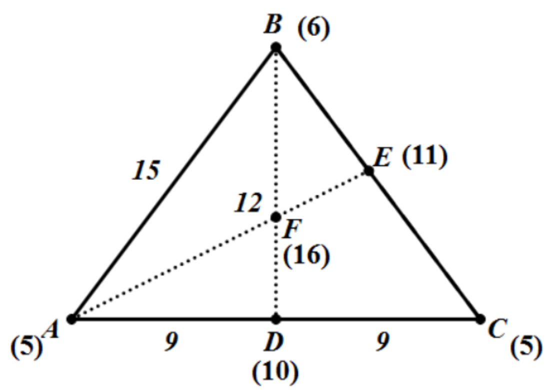
Evaluate the first few terms. $f(2) = \frac{1}{2}$, $f(3) = \frac{1}{6}$, $f(4) = \frac{1}{12}$, etc.
Notice that there's a pattern to how the terms can be expressed. For example, $f(2) = \frac{1}{2} = 1 - \frac{1}{2}$, $f(3) = \frac{1}{6} = \frac{1}{2} - \frac{1}{3}$, $f(4) = \frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, and so on. In general,

$\frac{1}{x(x-1)}$ can be expressed as $\frac{1}{x-1} - \frac{1}{x}$. When adding all the terms together, the middle terms cancel out, leaving the first and last terms.

$$1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} + ... - \cancel{\frac{1}{2016}} + \cancel{\frac{1}{2016}} - \frac{1}{2017}$$
$$1 - \frac{1}{2017} = \mathbf{\frac{2016}{2017}}$$

17. Isosceles triangle ABC with vertex B has a base of 18 and legs of 15. The median from point B intersects side AC at point D. The angle bisector from point A intersects side BC at point E. BD and AE intersect at point F. Find exactly the length of AF.

Solution: $\mathbf{\frac{9\sqrt{5}}{2}}$



Median BD is also the height. $BD = 12$ can be found through Pythagorean Theorem.
Next, we will apply the Angle Bisector Theorem along with mass points (the numbers in the parenthesis are masses of the points). We know that $\frac{AB}{AC} = \frac{5}{6}$, so $\frac{BE}{CE} = \frac{5}{6}$ according to the theorem. Assign masses to points B and C according to that ratio, and assign a mass to E, which is the sum of the masses of B and C. From those values we can also attain masses for points A, D, and F.

$$\frac{DF}{BF} = \frac{6}{10}$$
$$DF = \frac{12}{16} \times 6$$
$$DF = \frac{9}{2}$$

Lastly, use Pythagorean Theorem on ΔADF to find AF.

$$AF = \sqrt{9^2 + (\frac{9}{2})^2} = \sqrt{\frac{405}{4}} = \mathbf{\frac{9\sqrt{5}}{2}}$$

Alternative solution: locate a point G 9 units away from point A towards point B. $GB = 15 - 9 = 6$. Notice that ΔFDA is congruent to ΔFGA because of SAS. Thus, $\angle FGA = \angle FGB = 90°$. ΔBGF is a right triangle similar to ΔBDA. Now we can set up some proportions:

$$\frac{BD}{AB} = \frac{BG}{BF}$$
$$\frac{12}{15} = \frac{6}{BF}$$
$$BF = 7.5$$

$FD = 12 - 7.5 = 4.5$. Apply the Pythagorean Theorem on ΔFDA to get $AF = \mathbf{\frac{9\sqrt{5}}{2}}$.

18. The RMC Triathlon consists of a half-mile swim, an 18-mile bicycle ride, and a 5-mile run. Richard swims, bicycles, and runs at constant rates. He runs three times as fast as he swims, and he bicycles six times as fast as he runs. Richard completes the triathlon in three and a half hours. How many minutes does he spend swimming?

Solution: $\frac{630}{19}$ **minutes**

We use a table. Letting s represent the speed at which Richard swims at,

speed	time	distance
s		0.5
3s		5
18s		18

Thus, we if we add up the time it takes Richard to swim, the time it takes him to run, and the time it takes him to bike, we get the total time, or 3.5 hours. Since speed · time = distance,

$$\frac{0.5}{s} + \frac{5}{3s} + \frac{18}{18s} = 3.5$$

$$\frac{1}{s}(\frac{1}{2} + \frac{5}{3} + \frac{18}{18}) = 3.5$$
$$\frac{1}{s}(\frac{3}{6} + \frac{10}{6} + \frac{18}{18}) = 3.5$$
$$\frac{1}{s}(\frac{3}{6} + \frac{10}{6} + \frac{18}{18}) = 3.5$$

$$\frac{1}{s}(\frac{19}{6}) = \frac{7}{2}$$

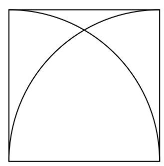
$$\frac{1}{s} = \frac{7}{2} \cdot \frac{6}{19} = \frac{21}{19}$$

$$s = \frac{19}{21}$$

Now that we have the speed, we can find the time it takes Richard to swim. time $= \frac{0.5}{\frac{19}{21}} = \frac{21}{38}$. This, however, is in hours. Thus, we multiply by 60 to convert to minutes.

$$\frac{21}{38} \cdot 60 = \frac{630}{19}$$

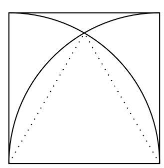
19.



In the diagram above, two circular arcs are inscribed inside a unit square. Find exactly the area of the union of the two arcs (the parabola-like shape in the middle).

Solution: $\boldsymbol{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}$

Connect the intersection point of the two arcs to the bottom vertices of the square. This creates an equilateral triangle with sides equal to 1 since those segments are radii of the circles.

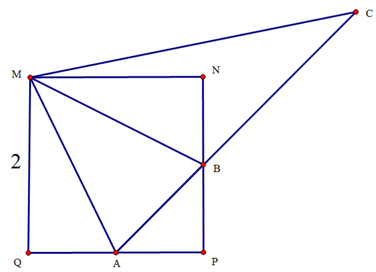


Because the angles of the triangle are 60°, each of the arc that makes up the parabola-like shape are 60° too. The area of one sector that contains the arc is $\frac{\pi}{6}$. To calculate the remaining area, we can take the area of the sector and subtract from it the area of the equilateral triangle: $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. Adding the two areas together, we get $\frac{\pi}{6} + \frac{\pi}{6} - \frac{\sqrt{3}}{4} = \boldsymbol{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}$.

20. Square $MNPQ$ has side length 2. Let A be the midpoint of PQ, and B the midpoint of NP. C is on AB such that B is between A and C, and $m\angle AMB = m\angle BMC$, where $AM \neq MC$. Compute exactly the length of MC.

Solution: $\boldsymbol{\frac{5\sqrt{5}}{3}}$

We begin by drawing the figure.



By the Pythagorean Theorem, $MA = MB = \sqrt{5}$ and $AB = \sqrt{2}$. Let $\theta = \angle AMB = \angle BMC$. By the Law of Cosines on $\triangle MAB$,

$$\left(\sqrt{2}\right)^2 = \left(\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 - 2 \cdot \left(\sqrt{5}\right) \cdot \left(\sqrt{5}\right) \cos\theta \Longrightarrow \cos\theta = \frac{4}{5}.$$

The Law of Cosines on $\triangle BMC$ yields
$BC^2 = MC^2 + \left(\sqrt{5}\right)^2 - 2 \cdot MC \cdot \left(\sqrt{5}\right) \cos\theta = MC^2 - \frac{8}{\sqrt{5}} MC + 5$. The Angle Bisector Theorem on $\triangle AMC$ yields

$$\frac{MC}{BC} = \frac{AM}{AB} = \sqrt{\frac{5}{2}} \Longrightarrow BC = \sqrt{\frac{2}{5}} MC.$$

Substituting,

$$0 = 3MC^2 - 8\sqrt{5}MC + 25$$
$$MC = \frac{8\sqrt{5} \pm \sqrt{20}}{6} = \frac{5\sqrt{5}}{3}.$$

Note that the answer of $\sqrt{5}$ does not work, as the problem condition states that $AM \neq MC$.

21. Given $a + \frac{1}{a} = 3$, find exactly:

$$\frac{a^6}{a^{12} + 1}$$

Solution: $\frac{1}{322}$

Let's set $\frac{a^6}{a^{12}+1} = N$. Then,

$$\frac{1}{N} = \frac{a^{12} + 1}{a^6} = a^6 + \frac{1}{a^6}$$

From here, we see that the RHS is in the form of $a^3 + \frac{1}{a^3}$. So,

$$a^6 + \frac{1}{a^6} = (a^2 + \frac{1}{a^2})(a^4 - 1 + \frac{1}{a^4})$$

We are given that $a + \frac{1}{a} = 3$. Thus,

$$(a + \frac{1}{a})^2 = a^2 + 2 + \frac{1}{a^2} = 3^2 = 9$$

So,

$$a^2 + \frac{1}{a^2} = 7$$

If we again square this,

$$(a^2 + \frac{1}{a^2})^2 = a^4 + 2 + \frac{1}{a^4} = 7^2 = 49$$

So,

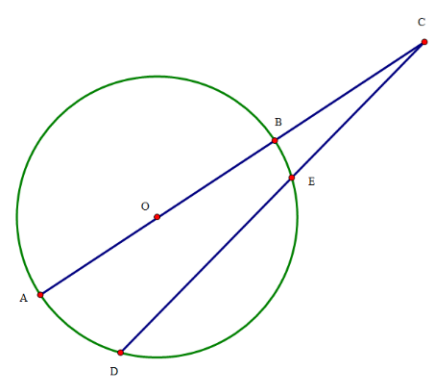
$$a^4 + \frac{1}{a^4} = 47$$

Now we can simply plug it into our original expression.

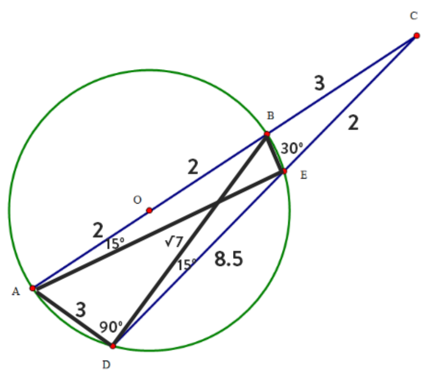
$$a^6 + \frac{1}{a^6} = (a^2 + \frac{1}{a^2})(a^4 - 1 + \frac{1}{a^4}) = (7)(46) = 322$$

However, don't forget that this is $\frac{1}{N}$! Our final answer is 1/322 or $\frac{1}{322}$.

22. In the figure above, line AC passes through the center of the circle O, hitting the circle at another point B. The radius of this circle is 2. Another secant line intersects the circle as shown, hitting the circle at E and D, where $CD > CE$. Given that $BC = 3$, $CE = 2$, $BD = \sqrt{7}$, and $\angle BAE = 15°$, calculate the exact area of triangle ΔAED.



Solution: $\frac{\mathbf{5(\sqrt{2}+\sqrt{6})}}{\mathbf{16}}$



Use power of a point to find the length of DE.

$$CB \times CA = CE \times CD$$
$$3 \times 7 = 2 \times CE$$
$$CE = 10.5$$
$$DE = 10.5 - 2 = 8.5$$

Notice that ΔADB is a right triangle since it's inscribed in a semicircle with the diameter as the hypotenuse. $\angle ADB = 90°$.
Arc $BE = 30°$ because inscribed angle $\angle BAE = 15°$. That means $\angle BDE = 15°$. Add $\angle BDE$ and $\angle ADB$ to get $\angle ADE = 105°$.
Use Pythagorean Theorem on ΔADB to get AD. $AD = \sqrt{4^2 - \sqrt{7}^2} = 3$.
Calculate the area of ΔAED using the formula Area$= \frac{1}{2}ab \sin c$.

$$[\Delta AED] = \frac{1}{2} \times AD \times DE \times \sin \angle ADE$$
$$[\Delta AED] = \frac{1}{2} \times 3 \times 8.5 \times \sin 105°$$

$$\sin 105° = \sin(45°+60°) = (\sin 45°)(\cos 60°)+(\sin 60°)(\cos 45°) = (\frac{\sqrt{2}}{2})(\frac{1}{2})+(\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$[\Delta AED] = \frac{1}{2} \times 3 \times 8.5 \times \frac{\sqrt{2}+\sqrt{6}}{4}$$
$$[\Delta AED] = (\frac{51}{4})(\frac{\sqrt{2}+\sqrt{6}}{4})$$
$$[\Delta AED] = \frac{\mathbf{5(\sqrt{2}+\sqrt{6})}}{\mathbf{16}}$$

23. Positive integers x and y satisfy the condition

$$\log_3(\log_{3^x}(\log_{3^y}(3^{3375}))) = 0.$$

Find the sum of all possible values of $x + y$.

Solution: 1631
Let's first simplify the expression.

$$(\log_{3^x}(\log_{3^y}(3^{3375}))) = 1$$

$$(\log_{3^y}(3^{3375})) = 3^x$$
$$3^{3375} = (3^y)^{3^x} = 3^{y3^x}$$

So,
$$3375 = 3^3 \cdot 5^3 = y3^x$$

Since y and x are positive integers, x must be $1, 2$, or 3.
When $x = 1$,
$$y = 3^2 \cdot 5^3 = 1125$$

When $x = 2$,
$$y = 3^1 \cdot 5^3 = 375$$

When $x = 3$,
$$y = 5^3 = 125$$

Thus, the sum of all possible $x + y$ is $1 + 2 + 3 + 1125 + 375 + 125 = 1631$.

24. Find the real number n such that

$$\arctan\frac{1}{4} + \arctan\frac{1}{5} + \arctan\frac{1}{6} + \arctan\frac{1}{n} = \frac{\pi}{4}.$$

Solution: $\frac{89}{16}$
Since we are dealing with acute angles, $\tan(\arctan a) = a$.Note that $\tan(\arctan a + \arctan b) = \frac{a+b}{1-ab}$, by tangent addition. Thus, $\arctan a + \arctan b = \arctan\frac{a+b}{1-ab}$.

Thu,s we can continuously apply this. Let's start with the first two terms. We get

$$\arctan\frac{1}{4}+\arctan\frac{1}{5}=\arctan\frac{9}{19}$$

.
Now,

$$\arctan\frac{9}{19}+\arctan\frac{1}{6}=\arctan\frac{73}{105}$$

.
We now have

$$\arctan\frac{73}{105}+\arctan\frac{1}{n}=\frac{\pi}{4}=\arctan 1$$

Thus,

$$\frac{\frac{73}{105}+\frac{1}{n}}{1-\frac{73}{105n}}=1$$

Simplifying,

$$\frac{73}{105}+\frac{1}{n}=1-\frac{73}{105n}$$
$$\frac{1}{n}+\frac{73}{105n}=1-\frac{73}{105}$$
$$\frac{1}{n}(1+\frac{73}{105})=\frac{32}{105}$$
$$\frac{1}{n}(\frac{178}{105})=\frac{32}{105}$$
$$\frac{1}{n}=\frac{32}{105}\cdot\frac{105}{178}$$
$$\frac{1}{n}=\frac{32}{178}=\frac{16}{89}$$
$$n=\frac{89}{16}$$

25. The points $(0,0)$, $(m,8)$, and $(n,36)$ are the vertices of an equilateral triangle. Find the value of mn.

Solution: $\frac{1280}{3}$
Consider the points on the complex plane. The point $n+36i$ is then a rotation of 60 degrees of $m+8i$ about the origin, so:

$$(m+8i)(\operatorname{cis}60^\circ)=(m+8i)\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)=n+36i.$$

Equating the real and imaginary parts, we have:

$$n = \frac{m}{2} - \frac{8\sqrt{3}}{2}$$
$$36 = \frac{8}{2} + \frac{m\sqrt{3}}{2}$$

Solving this system, we find that $m = \dfrac{64\sqrt{3}}{3}, n = \dfrac{20\sqrt{3}}{3}$. Thus, the answer is $\dfrac{1280}{3}$.

26. Find the minimum value of $\frac{16x^2\cos^2 x+9}{x\cos x}$ for $0 < x < \frac{\pi}{2}$.

Solution: 24
Set $a = x\cos x$ Then,

$$\frac{16x^2\cos^2 x + 9}{x\cos x} = \frac{16a^2+9}{a} = 16a + \frac{9}{a}$$

Applying AM-GM,

$$\frac{16a + \frac{9}{a}}{2} \geq \sqrt{16a \cdot \frac{9}{a}} \geq \sqrt{16 \cdot 9} \geq \sqrt{144} \geq 12$$

So,
$$16a + \frac{9}{a} \geq 24$$

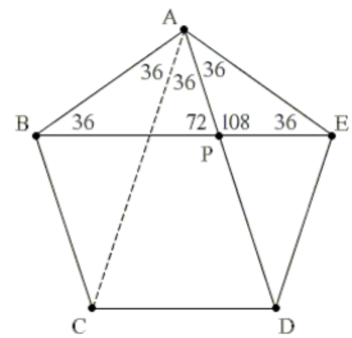
27. In the land of Richardtopia, the currency is measured in T coins. There are only two coins in this society: a 69 T coin, and a 343 T coin. The number ”awesomeness” is an important number in this society, as it is the largest value that cannot be made by the currency. Find this number.

Solution: 23255
The Chicken McNugget Theorem comes in handy here. The theorem states that given any two relatively prime positive integers m and n, the greatest integer the two numbers cannot make is $mn - m - n$. So, in this case, that's $343 * 69 - 343 - 60$, or 23255.

28. Calculate the exact value of $\sin(36)^\circ$.

Solution: $\frac{\sqrt{10-2\sqrt{5}}}{4}$

We will use a regular pentagon to first find $\cos(36°)$.



A regular pentagon can be inscribed in a circle, and its vertices are equally spaced on the circumference. $\angle CAD = \frac{360°}{5}/2 = 36°$. From there, we can use isosceles triangles and symmetry to calculate other angles as shown above.
Notice that ΔABE is similar to ΔPEA.

$$\frac{BE}{AB} = \frac{AE}{EP}$$
$$AE = AB, BE \times EP = AB^2$$
$$(BP + EP) \times EP = AB^2$$
$$AB = BP, (AB + EP) \times EP = AB^2$$

If we assign length 1 to EP and set AB equal to x, the ratio $\frac{AB}{EP}$ becomes x. Plug those values into the previous equation, we get

$$x + 1 = x^2$$
$$or$$
$$x^2 - x - 1 = 0$$

Solve that using the quadratic equation. We get a positive solution of $x = \frac{1+\sqrt{5}}{2}$.
This is known as the golden ratio.
Because AB=AE, the ratio $\frac{AE}{EP}$ is also x. Drop an altitude from P to AE, and we can establish

$$\cos(\angle AEP) = \frac{\frac{AE}{2}}{EP} = \frac{\frac{AE}{EP}}{2} = \frac{x}{2} = \frac{1+\sqrt{5}}{4}$$
$$\angle AEP = 36°, \cos(36°) = \frac{1+\sqrt{5}}{4}$$

We can then calculate $\cos(18°)$ using $\cos 2\alpha = 2\cos^2\alpha - 1$, where $\alpha = 18°$. $\cos(18°) = \frac{\sqrt{10+2\sqrt{5}}}{4}$.

Use $\cos^2\alpha + \sin^2\alpha = 1$ to get that $\sin(18°) = \frac{\sqrt{5}-1}{4}$.
Lastly, use $\sin 2\alpha = 2\sin\alpha\cos\alpha$ to calculate $\sin(36°)$, which is $\frac{\sqrt{10-2\sqrt{5}}}{4}$.

29. There is one real root in the polynomial $17x^3 - 12x^2 - 6x - 1 = 0$. Calculate it exactly.

Solution: $\dfrac{5\sqrt[3]{5} + 2\sqrt[3]{25} + 4}{17}$
We must first notice the coefficients of the polynomial.

$$17x^3 - 12x^2 - 6x - 1 = 0$$

$$17x^3 = 12x^2 + 6x + 1$$

The coefficients on the right-hand side (RHS) are in the expansion of $(2x+1)^3$!!!.

$$(2x+1)^3 = 8x^3 + 12x^2 + 6x + 1$$

Adding $8x^3$ to both sides of the original equation,

$$25x^3 = 8x^3 + 12x^2 + 6x + 1$$

$$25x^3 = (2x+1)^3$$

Cube rooting both sides,

$$\sqrt[3]{25}x = 2x + 1$$

$$\sqrt[3]{25}x - 2x = 1$$

$$x(\sqrt[3]{25} - 2) = 1$$

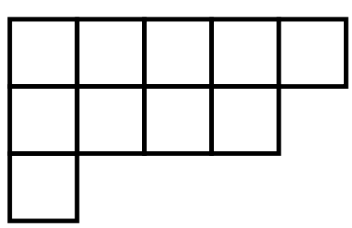
$$x = \frac{1}{\sqrt[3]{25} - 2}$$

The denominator can be simplified through the algebraic manipulation of $a^3 - b^3$, to get rid of the radical in the denominator.

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$x = \frac{1}{\sqrt[3]{25} - 2} \cdot \frac{\sqrt[3]{625} + 2\sqrt[3]{25} + 4}{\sqrt[3]{625} + 2\sqrt[3]{25} + 4} = \frac{\sqrt[3]{625} + 2\sqrt[3]{25} + 4}{17} = \frac{5\sqrt[3]{5} + 2\sqrt[3]{25} + 4}{17}$$

30.



How many distinct ways can we arrange the numbers 1 to 10 into the boxes in the figure above so that, when going right, the numbers in each row are strictly increasing and, when going down rows, the numbers in each column are strictly increasing?

Solution: 288
There is an amazing formula for this problem, also known as the hook-length in a young tableau. For more information on this, please look at `https://en.wikipedia.org/wiki/Young_tableau#Dimension_of_a_representation`

$$d_y = \frac{10!}{7 * 5 * 4 * 3 * 5 * 3 * 2 * 1 * 1} = 288$$

Otherwise, gruesome casework also leads to an equivalent solution.