Rochester Math Club Placement Test Solutions (Intermediate)

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1 Solutions

1. Find the value of x in the equation $3x + 8 = 32$.

Solution: 8

$$3x + 8 = 32$$
$$3x = 24$$
$$x = 8$$

2. What percent of 170 is 34?

Solution: 20%
The question is asking how much of 170 is 34, or, what the value of $\frac{34}{170}$ is.

$$\frac{34}{170} = \frac{2}{10} = \frac{1}{5} = 0.2 = 20\%$$

3. Find the smallest integer value of x that satisfies the inequality:

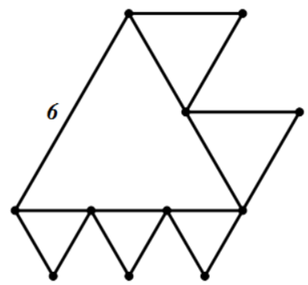
$$2(x - 1) > -3(x + 5)$$

Solution: -2

$$2x - 2 > -3x - 15$$
$$5x > -13$$
$$x > -\frac{13}{5} > -2.6$$

Thus, the smallest integer greater than -2.6 is -2.

4. All of the triangles in the figure below are equilateral. Find the area of the entire figure.



Solution: $\mathbf{\frac{33\sqrt{3}}{2}}$

The area of an equilateral triangle is $s^2\frac{\sqrt{3}}{4}$, where s is the side length of the triangle.

The smallest triangles have side length of $6/3 = 2$. Since there are three of them, the total area of the smallest triangles is $3 \cdot (2^2\frac{\sqrt{3}}{4}) = 3\sqrt{3}$.
The middle-size triangles have side length of $6/2 = 3$. Since there are two of them, the total area of the middle triangles is $2 \cdot (3^2\frac{\sqrt{3}}{4}) = \frac{9\sqrt{3}}{2}$. The largest triangle has side length 6. So, the area of this triangle is $6^2\frac{\sqrt{3}}{4} = 9\sqrt{3}$.

Summing,

$$3\sqrt{3} + \frac{9\sqrt{3}}{2} + 9\sqrt{3} = \frac{33\sqrt{3}}{2}$$

5. Eight years ago, Tracy's age was one-fifth of her mom's age. Sixteen years later, her age will be half of her mom's age. How old is she this year?

Solution: 16
Let's set Tracy's age this year as T and her mom's age this year as M. We have a system of equations:

$$5(T - 8) = M - 8$$
$$2(T + 16) = M + 16$$

Proceed to solve this system.

$$\begin{aligned} M &= 5T - 32 \\ M &= 2T + 16 \\ 5T - 32 &= 2T + 16 \\ 3T &= 48 \\ T &= 16 \end{aligned}$$

6. A cylinder has a radius of 4. The ratio of the radius to the height of the cylinder is 2 : 5. Find the volume of the cylinder.

Solution: 160π
Since the radius is 4 and the ratio of it to the height is 2 : 5, multiply 4 by $\frac{5}{2}$ to get the height which equals 10. The volume of the cylinder is $(4^2)\pi(10) = 160\pi$.

7. Given that $a * b = 2a + b^2$, find x if $x * (2 * 3) = 201$.

Solution: 32
Begin by calculating what's inside of the parenthesis.

$$2 * 3 = (2)(2) + 3^2 = 13$$

Then use 13 to find x.

$$\begin{aligned} 2x + 13^2 &= 201 \\ 2x &= 32 \\ x &= 16 \end{aligned}$$

8. $f(x)$ is a linear function with a positive slope. If the slope of $f(f(x))+f(x)$ is 20, find the slope of $f(x)$.

Solution: 4
Since $f(x)$ is a linear function, it has the form of $mx + b$ where m is the slope. $f(f(x))$ will take the form $m(mx+b)+b$, which simplifies down to $m^2x+mb+b$. Add $f(x)$ and we get $(m^2x + mb + b) + (mx + b)$. Combine all the like terms, and we have $(m^2 + m)x + (mb + 2b)$. $m^2 + m$ is the slope.

$$\begin{aligned} m^2 + m &= 20 \\ m^2 + m - 20 &= 0 \\ (m + 5)(m - 4) &= 0 \\ m &= -5, 4 \end{aligned}$$

Since the question asks for a positive slope, the answer is **4**.

9. What is the probability that the sum of two randomly selected factors of 96 is odd?

Solution: $\mathbf{\frac{1}{6}}$
Start by listing the factors of 96, which are $1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96$. There are 12 factors in total. The odd factors are $1, 3$, so there are 2 odd factors. The probability is thus $\frac{2}{12} = \mathbf{\frac{1}{6}}$.

10. 132 in base b is equal to 72 in base 10. Find b.

Solution: 7
132 in base b can be expressed as $b^2 + 3b + 2$. We can have a quadratic equation by setting that equal to 72.

$$\begin{aligned} b^2 + 3b + 2 &= 72 \\ b^2 + 3b - 70 &= 0 \\ (b + 10)(b - 7) &= 0 \\ b &= -10, 7 \end{aligned}$$

Since a base must be greater than 0, the answer is **7**.

11. Tracy wants to make a playlist of awesome music. There has to be at least one rap song and one rock song. She can choose from 5 rap songs, 3 rock songs, and 4 pop songs. Playlists with the same songs but different orders are considered the same. How many different playlists of 4 songs can she make?

Solution: 345
Let's make a table with the possible cases.Remember to consider the combinations within a genre.

Rap	Rock	Pop	Combinations
3	1	0	$\binom{5}{3}\binom{3}{1} = 30$
2	2	0	$\binom{5}{2}\binom{3}{2} = 30$
1	3	0	$\binom{5}{1}\binom{3}{3} = 15$
2	1	1	$\binom{5}{2}\binom{3}{1}\binom{4}{1} = 120$
1	2	1	$\binom{5}{1}\binom{3}{2}\binom{4}{1} = 60$
1	1	2	$\binom{5}{1}\binom{3}{1}\binom{4}{2} = 90$

Add all of the combinations to get **345**.

12. S_1 is a geometric series, and S_2 is an arithmetic series with 10 terms. The common ratio of the terms in S_1 is equal to the opposite of the common difference of the terms in S_2. The first term of S_1 is 3, and the fourth term is -24. Find S_2 if its first term is 2.

Solution: 120

Since S_1 is 3 and S_4 is -24, we can establish that $-24 = 3 \times r^3$, where r is the common ratio. Solve that equation to get $r = -2$. The common difference of terms in S_2 is then 2. To find the sum of 10 terms, we need to first get the last term: $3 + 2(10 - 1) = 21$. Use the summation formula to get the sum:

$$\frac{(3+21)(10)}{2} = 120$$

13. Determine exactly the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + ...}}}$

Solution: 3

Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + ...}}}$
Squaring,

$$x^2 = 6 + \sqrt{6 + \sqrt{6 + ...}}$$

But

$$x = \sqrt{6 + \sqrt{6 + \sqrt{6 + ...}}}$$

So

$$x^2 = 6 + x$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$

So

$$x = 3,$$

as $x = -2$ is extraneous.

14. If the sides of a triangle have lengths $4, 5$, and 6, what is the radius of the circle circumscribing the triangle?

Solution: $\mathbf{\frac{8\sqrt{7}}{7}}$
The area of a triangle is equal to the product of its sides divided by the four times the radius of the circumcircle, or:

$$A = \frac{abc}{4R}$$

where a, b, c are the sides of the triangle, R is the radius of the circumscribing circle, and A is the area of the triangle. The area of the triangle can be found by Heron's formula, or

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where the semi-perimeter $s = \frac{a+b+c}{2}$.

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{15}{2}(\frac{15}{2}-4)(\frac{15}{2}-5)(\frac{15}{2}-6)}$$

$$= \sqrt{\frac{15}{2}(\frac{7}{2})(\frac{5}{2})(\frac{3}{2})}$$

$$= \frac{15\sqrt{7}}{4}$$

$$\frac{15\sqrt{7}}{4} = \frac{4 \cdot 5 \cdot 6}{4R}$$

$$15\sqrt{7} = \frac{4 \cdot 5 \cdot 6}{R}$$

$$15R\sqrt{7} = 4 \cdot 5 \cdot 6$$

$$3R\sqrt{7} = 4 \cdot 6$$

$$R\sqrt{7} = 4 \cdot 2 = 8$$

$$R = \frac{8}{\sqrt{7}}$$

$$R = \frac{8\sqrt{7}}{7}$$

15. Determine exactly the value of:

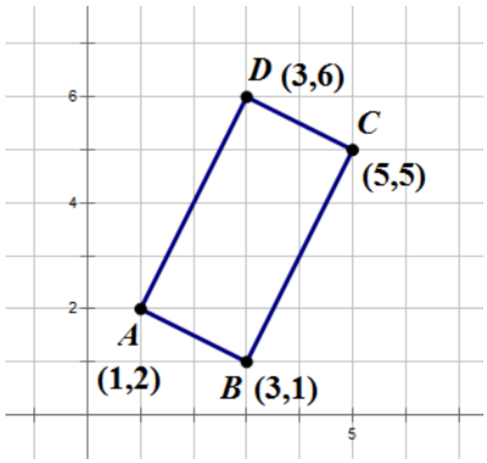
$$\frac{2}{\log_{12} 4} - \frac{1}{\log_{18} 4} + \frac{1}{log_8 4}$$

Solution: 0
We use a logarithmic property. Given any $\log_a b$, $\frac{1}{\log_a b} = \log_b a$. So, this simply becomes:

$$2\log_4 12 - \log_4 18 - \log_4 8 = \log_4 144 - \log_4 18 - \log_4 8 = \log_4 \frac{144}{18 \cdot 8} = \log_4 1 = 0$$

16. Rectangle $ABCD$ in the first quadrant of the coordinate plane has the given coordinates $A(1,2)$ and $B(3,1)$. BC has the length of $\sqrt{20}$. The area of $ABCD$ is w. The coordinates of C and D can be expressed as (m,n) and (x,y) respectively. Find $w + m + n - x - y$.

Solution: 11



The slope of AB is $\frac{1}{2}$, so the slope of BC is 2 because it is perpendicular to AB. The line equation for BC is $y = 2x - 5$. We can use the distance formula to get the coordinates of C.

$$\begin{aligned}
\sqrt{20} &= \sqrt{(x-3)^2 + (2x-5-1)^2} \\
20 &= (x-3)^2 + (2x-6)^2 \\
20 &= x^2 - 6x + 9 + 4x^2 - 24x + 36 \\
5x^2 - 30x + 25 &= 0 \\
x^2 - 6x + 5 &= 0 \\
(x-1)(x-5) &= 0 \\
x &= 1, 5
\end{aligned}$$

5 is the answer that satisfies C being in the first quadrant, and plugging that into the line equation, we get the coordinates of $C = (5, 5)$. Using the slope of AB, we can deduce that D has the coordinates $(3, 6)$. Use the distance formula again to calculate the distance of AB.

$$\sqrt{(3-1)^2 - (2-1)^2} = \sqrt{5}$$

The area of $ABCD = \sqrt{20}\sqrt{5} = 10$. $10 + 5 + 5 - 3 - 6 = \mathbf{11}$.

17. α is an acute angle such that $\sin\alpha = \sqrt{\frac{2x+1}{5}}$. Find exactly $\tan 2\alpha$.

Solution: $\mathbf{\frac{2\sqrt{6x-4x^2+4}}{5-4x}}$

Square both sides to get $\sin^2\alpha = \frac{2x+1}{5}$. We can then use the Pythagorean identity to get $\cos^2\alpha$, which is $1 - \sin^2\alpha = 1 - \frac{2x+1}{5} = \frac{4-2x}{5}$. $\cos\alpha =$

$\sqrt{\frac{4-2x}{5}}$. We can then use the double-angle formulas to get $\sin 2\alpha$ and $\cos 2\alpha$. $\sin 2\alpha = 2(\sqrt{\frac{2x+1}{5}})(\sqrt{\frac{4-2x}{5}})$, which simplifies down to $\frac{2\sqrt{6x-4x^2+4}}{5}$.$\cos 2\alpha = \frac{4-2x}{5} - \frac{2x+1}{5} = \frac{5-4x}{5}$. $\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sqrt{6x-4x^2+4}}{5-4x}$.

18. Determine exactly $6x_3 + 3x_5$ if x_1, x_2, x_3, x_4, and x_5 satisfy the system of equations below.

$$3x_1 + x_2 + x_3 + x_4 + x_5 = 4$$

$$x_1 + 3x_2 + x_3 + x_4 + x_5 = 15$$

$$x_1 + x_2 + 3x_3 + x_4 + x_5 = 22$$

$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 54$$

$$x_1 + x_2 + x_3 + x_4 + 3x_5 = 72$$

Solution: $\mathbf{\frac{693}{14}}$

Summing all the equations, we get that

$$7x_1 + 7x_2 + 7x_3 + 7x_4 + 7x_5 = 277$$

and

$$x_1 + x_2 + x_3 + x_4 + x_5 = \frac{277}{7}$$

Now, if we subtract this equation from each of the original equations, we can find each value. However, since the problem only wants the value of x_3 and x_5, we subtract the third and fifth equations from the one we just got.

$$2x_3 = 22 - \frac{277}{7} = -\frac{123}{7} = -\frac{123}{14}$$

$$2x_5 = 72 - \frac{277}{7} = \frac{477}{7} = \frac{477}{14}$$

$$6(-\frac{123}{14}) + 3(\frac{477}{14}) = \frac{693}{14}$$

19. What is the remainder when $2^0 + 2^1 + 2^2 + \cdots + 2^{69421}$ is divided by 7?

Solution: 1

We notice that the sum of of every three terms (first, second, third, then fourth, fifth, sixth) sum up to a multiple of 7, leaving a remainder of 0.

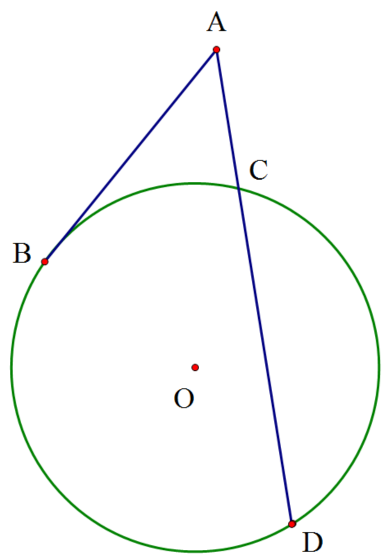
$$2^0 + 2^1 + 2^2 = 7 \equiv 0 \mod 7$$

$$2^3 + 2^4 + 2^5 = 56 \equiv 0 \mod 7$$

$$2^4 + 2^5 + 2^6 = 112 \equiv 0 \mod 7$$

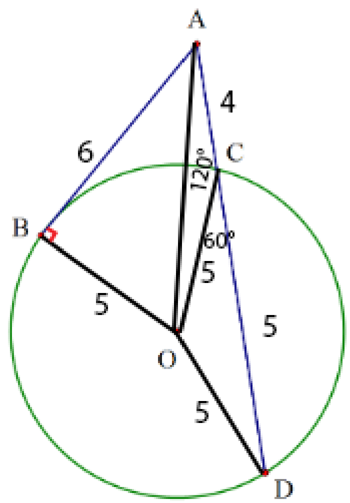
So, we just need to find when the cycles of 3 ends. Since $69420 \equiv 0 \mod 3$ we focus on 2^{69421}.
The first term in each set of three is always $1 \mod 7$.
The second term in each set of three is always $2 \mod 7$.
The third term in each set of three is always $4 \mod 7$.
Thus, since 2^{69421} is the first term in the next set of three, the remainder is simply 1.

20.



In the figure above, segment AB is tangent to circle O at B. Given that that $OB = 5$, $BA = 6$, and $AC = 4$, find exactly the area of the quadrilateral $ABOC$.

Solution: $\mathbf{15 + 5\sqrt{3}}$



Using power of a point, we have the equation:

$$\begin{aligned} (AB)^2 &= (AC)(AD) \\ 36 &= 4(AD) \\ AD &= 9 \\ CD &= 9 - 4 = 5 \end{aligned}$$

OB is the radius and perpendicular to AB. ΔABO is a right triangle with area $(\frac{1}{2})(5)(6) = 15$.
Connect OC and OD. Since they are both radii with length of 5, and CD is also 5, ΔCOD is an equilateral triangle. $\angle OCD = 60°$, $\angle ACO = 120°$. We can calculate the area of ΔACO by using $Area\Delta = \frac{1}{2}ab\sin c = \frac{1}{2}(4)(5)(\sin 120°) = 5\sqrt{3}$.
Add the two areas to get the area of quadrilateral $ABOC = \mathbf{15 + 5\sqrt{3}}$.

21. Find the least positive integer greater than $(\sqrt{5} + \sqrt{3})^6$.

Solution: 3904
Let $x = \sqrt{5} + \sqrt{3}$ and $y = \sqrt{5} - \sqrt{3}$.

Then,

$$x + y = 2\sqrt{5}$$
$$xy = 2$$

So,

$$x^2 + y^2 = (x + y)^2 - 2xy = 20 - 4 = 16$$

$$x^6+y^6 = (x^2+y^2)^3-3x^2y^2(x^2+y^2) = 16^3-3(4)(16) = 16(256-12) = 16(244) = 3904$$

$$(\sqrt{5}+\sqrt{3})^6 + (\sqrt{5}-\sqrt{3})^6 = 3904 - (\sqrt{5}-\sqrt{3})^6$$

But we know that $0 < \sqrt{5}-\sqrt{3} << 1$.
So, we must subtract a tiny tiny amount from 3904, so the smallest integer greater than $(\sqrt{5}+\sqrt{3})^6$ is **3904**. (The real value is actually $3903.9836064887\cdots$.)

22. Find exactly the largest possible distance between two points, one on the sphere of radius 15 with center $(-7,-12,12)$ and the other on the sphere of radius 55 with center $(18,12,-16)$.

Solution: $70+\sqrt{1985}$
The line connecting the largest possible distance between two points on the spheres goes through the center. In fact, the largest distance the the sum of the two radii and the distance between the centers of the circle. This is simply:

$$15+55+\sqrt{(18-(-7))^2+(12-(-12))^2+(-16-12)^2}$$

$$= 70+\sqrt{625+576+784} = 70+\sqrt{1985}$$

23. Richard and Tracy sprint for 60 minutes on a circular track. Richard runs clockwise at 200 meters per minute and uses the inner lane of the track that has radius 50 meters. Tracy runs counterclockwise at 240 m/min and uses the outermost lane with radius of 60 meters. She starts on the same radial line as Richard. After they start, how many times do they pass each other?

Solution: 76
Since $d = rt$, we note that Richard runs one lap in $\frac{2\cdot 50\pi}{200} = \frac{\pi}{2}$ minutes, while Tracy also runs one lap in $\frac{2\cdot 60\pi}{240} = \frac{\pi}{2}$ minutes. They take the same amount of time to run a lap, and since they are running in opposite directions they will meet exactly twice per lap (once at the starting point, the other at the half-way point). Thus, there are $\frac{60}{\frac{\pi}{2}} \approx 38.2$ laps run by both, or $\lfloor 2\cdot 38.2\rfloor = 76$.

24. An unfair eight-sided die with faces numbered $1-8$ has the properties that when rolled, the probability of obtaining face F is less than $\frac{1}{8}$, the probability of obtaining the face opposite of F is greater than $\frac{1}{8}$, and the probability of obtaining the six remaining faces is $\frac{1}{8}$. The sum of numbers on opposite faces is 9. When two such dies are rolled, the probability of obtaining a sum of 9 is $\frac{79}{512}$. What is the probability of obtaining face F?

Solution: $\frac{3}{32}$
Let $\frac{1}{8}-x$ be the probability of obtaining face F. Thus, the probability of obtaining the face opposite of F is $\frac{1}{8}+x$. Add all the probabilities of obtaining a

sum of 9 to set up an equation.

$$(2)(2)(2)((\frac{1}{8})^2) + (2)(\frac{1}{8} - x)(\frac{1}{8} + x) = \frac{79}{512}$$

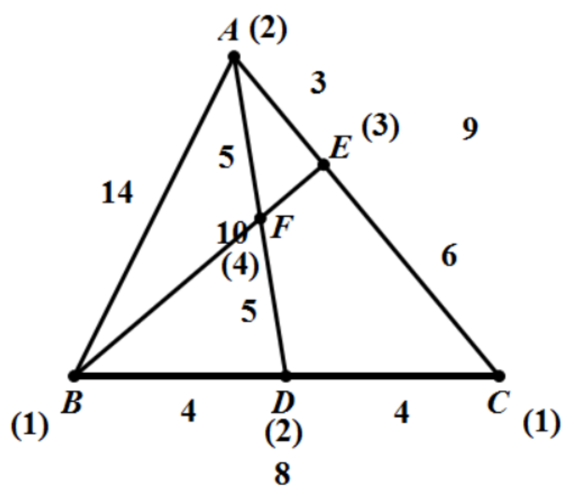
We know that a sum of 9 can be obtained through rolling two opposite faces, and for each case of rolling two opposite faces, there are be two permutations. (Ex: rolling 1 then 8 and rolling 8 then 1 are two permutations). Thus, we multiply 2^3 for the three pairs of opposite faces which have the probability of $\frac{1}{8}$ for obtaining each face, and 2 for the unfair pair. We proceed to solve the equation:

$$\begin{aligned}
\frac{8}{64} + (2)(\frac{1}{64} - x^2) &= \frac{79}{512} \\
\frac{1}{8} + \frac{1}{32} - 2x^2 &= \frac{79}{512} \\
64 + 16 - 1024x^2 &= 79 \\
1 &= 1024x^2 \\
x &= \frac{1}{32}
\end{aligned}$$

The probability of obtaining face F is $\frac{1}{8} - \frac{1}{32} = \mathbf{\frac{3}{32}}$.

25. Triangle ABC has side lengths $AC = 9$ and $BC = 8$. A cevian from B intersects AC at E. The median from A intersects BC at D and has a length of 10. AD and BE intersect at F, and $BF : FE = 3 : 1$. Find exactly $\cos(\angle AFB)$.

Solution: $\mathbf{\frac{303}{280}}$



Start by labeling points with their masses (numbers in parenthesis are masses). Because AD is the median, we have $B = (1), C = (1), D = (2)$. We know BF and FE have a ratio of $3 : 1$, so we label $E = 3$. F has a mass of (4), obtained by summing the masses of B and E. Notice that A has a mass of 2 by subtraction, which means AD is bisected into lengths of 5 and 5. Use Stewart's formula to get the length of AB.

$$(AB)^2(6) + (8)^2(3) = (10)^2(9) + (9)(3)(6)$$
$$AB = \sqrt{145}$$

Use Stewart's again to find the length of BE.

$$(145)(6) + (64)(3) = (BE)^2(9) + (9)(3)(6)$$
$$BE = 10$$

$BE = \frac{10}{4}(3) = \frac{15}{2}$. Use law of cosines on ΔABF to find $\cos\angle AFB$.

$$5^2 = 14^2 + (\frac{15}{2})^2 - (2)(14)(\frac{15}{2})\cos\angle AFB$$
$$\cos\angle AFB = \frac{303}{280}$$

26. For all positive integers m, let $f(m) = \log_{2261} m^2$. Let $X = f(7) + f(17) + f(19)$. Find exactly the range of values X can take.

Solution: [2]
This is actually a very simple problem: Let's just plug in each value.

$$f(7) = 2\log_{2261} 7$$
$$f(17) = 2\log_{2261} 17$$
$$f(19) = 2\log_{2261} 19$$

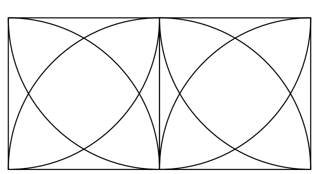
Adding,

$$f(7) + f(17) + f(19) = 2\log_{2261} 7 + 2\log_{2261} 17 + 2\log_{2261} 19$$

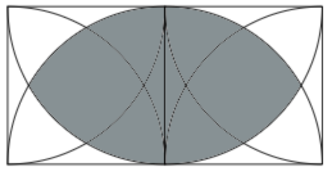
Using the property of logarithms ($\log_a b + \log_a c = \log_a bc$)

$$2\log_{2261} 7 + 2\log_{2261} 17 + 2\log_{2261} 19 = 2(\log_{2261} 2261) = 2(1) = 2$$

27.



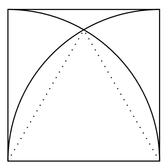
Two identical squares are put together in the diagram above. Each square has side length 2, with four quarter circles inscribed within the square (each quarter circle is centered at each of the vertices of the square).



A portion of the figure is now shaded. Calculate the area of the shaded region.

Solution: $\mathbf{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$

Connect the intersection point of the two arcs to the bottom vertices of the square. This creates an equilateral triangle with sides equal to 1 since those segments are radii of the circles.



Because the angles of the triangle are 60°, each of the arc that makes up the parabola-like shape are 60° too. The area of one sector that contains the arc is $\frac{\pi}{6}$. To calculate the remaining area, we can take the area of the sector and subtract from it the area of the equilateral triangle: $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. Adding the two areas together, we get $\frac{\pi}{6} + \frac{\pi}{6} - \frac{\sqrt{3}}{4} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$. Since there are two of these regions, we double the answer to get $\mathbf{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$.

28. How many sequences of 10 characters consist entirely of R's and T's and that have the property that every consecutive run of R's has even length, and every consecutive run of T's has an odd length? For example, sequences RR, T, and $RRTRR$ are valid, but $TTRT$ is not.

Solution: 80

Let r_n and t_n denote, respectively, the number of sequences of length n ending in R and T. If a sequence ends in an R, then it must have been formed by appending two Rs to the end of a string of length $n-2$. If a sequence ends in a T, it must have either been formed by appending one T to a string of length $n-1$ ending in an R, or by appending two Ts to a string of length $n-2$ ending in a T. Thus, we have the recursions

$$r_n = r_{n-2} + t_{n-2}$$
$$t_n = r_{n-1} + t_{n-2}$$

By counting, we find that $r_1 = 0, t_1 = 1, r_2 = 1, t_2 = 0$.

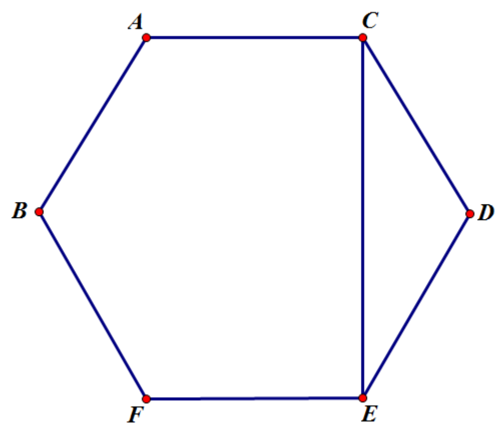
n	r_n	t_n	n	r_n	t_n
1	0	1	8	6	10
2	1	0	9	11	11
3	1	2	10	16	21
4	1	1	11	22	27
5	3	3	12	37	43
6	2	4			
7	6	5			

Therefore, the number of such strings of length 12 is $r_{12} + t_{12} = 80$.[2]

29. Compute exactly the area of the polygon whose vertices are the solutions in the complex plane to the polynomial $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

Solution: $\sqrt{3}$

The polynomial $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ is equivalent to $\dfrac{x^6 - 1}{x - 1}$, where $x \neq 1$. This is a regular hexagon in the complex plane–except it has one part of it chopped off. The figure below shows the situation.



In this case, triangle CDE is cut off, as $x \neq 1$, as it is an extraneous solution to the polynomial. Thus, our goal is to simply calculate the area of the hexagon minus the area of the triangle. We know that the distance from the center of the hexagon (origin) to a vertice is 1 (for example, $x = -1$ draws out a line to point B). Thus, the area of the hexagon is $\frac{3\sqrt{3}}{2}$. The area of triangle CDE can be found by $\dfrac{1}{2} \cdot CD \cdot DE \cdot \sin CDE$, heron's, and also by drawing down a height

[2] Retrieved from the AMC Competition Series

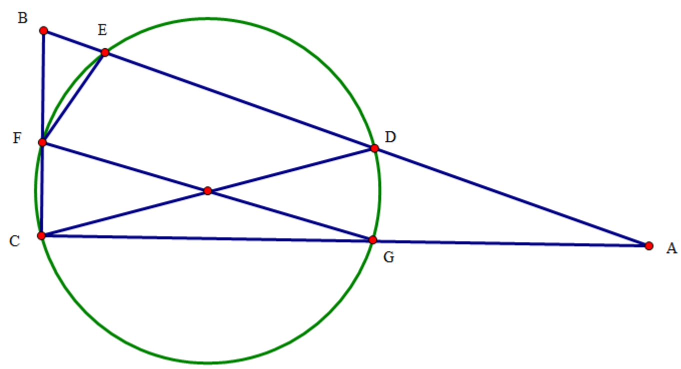
from D. We will only show one way here.

$$\frac{1}{2} \cdot CD \cdot DE \cdot \sin CDE = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 120 = \frac{\sqrt{3}}{2}$$

Thus, the area we want is simply

$$\frac{3\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \sqrt{3}$$

30.



In the figure above, CD is the diameter of the circle. Triangle ABC has right angle at C. Given that D is the midpoint of AE, $BC = 4\sqrt{5}$, $AC = 2$, and E, F, G all lie on the circle, is $m\angle B = m\angle EFG$? In addition, calculate exactly the length of the diameter of the circle.

Solution: Yes; $\boldsymbol{\frac{9\sqrt{21}}{21}}$
We first prove that $m\angle B = m\angle EFG$. We see that FG is also the diameter of the circle, because since $\angle C = 90°$, ΔFCG is a right triangle, and thus, FG is the diameter.

$$\angle B = \frac{\widehat{DGC} - \widehat{FE}}{2}$$

Similarly,

$$\angle EFG = \frac{\widehat{FEG} - \widehat{FE}}{2}$$

But

$$\widehat{DGC} = \widehat{FEG} = 180$$

So, $m\angle B = m\angle EFG$. Now we continue. Drop a perpendicular from C. The foot of the altitude actually hits E, because CD is a diameter. Now, since we are given two sides, we can find the hypotenuse, and thus the altitude. In addition, we can find the length of EA through the Pythagorean Theorem, and thus the length of ED, as D is the midpoint of the segment. After that, we can simply find the length of CD. Now we get into the math portion of the problem.

We can see that CE is the altitude of triangle ABC, as it makes a right angle (triangle CED has a hypotenuse as the diameter of the circle, and is thus a right triangle).
CE is equal to the product of the bases divided by the length of the hypotenuse. The hypotenuse length is $\sqrt{(4\sqrt{5})^2+4} = \sqrt{80+4} = \sqrt{84} = 2\sqrt{21}$ So, $CE = \frac{4\sqrt{5}\cdot 2}{2\sqrt{21}} = \frac{4\sqrt{5}}{\sqrt{21}}$. Let us call the length $DE = x$. Then, since D is the midpoint of AE, $AD = DE = x$. In addition,

$$AE^2 + CE^2 = 4$$

$$(2x)^2 + (\frac{4\sqrt{5}}{\sqrt{21}})^2 = 4$$

$$4x^2 + \frac{80}{21} = 4$$

$$4x^2 = \frac{4}{21}$$

$$x^2 = \frac{1}{21}$$

We can finally find the diameter of this circle.

$$DE^2 + CE^2 = CD^2$$

$$\frac{1}{21} + \frac{80}{21} = CD^2$$

$$\frac{81}{21} = CD^2$$

$$\frac{9\sqrt{21}}{21} = CD$$