Rochester Math Club Placement Test Solutions (Novice)

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1 Solutions

1. What is $1 + 2 + 3 \cdots + 8 + 9 + 10$?

Solution: 55
You can choose to add all of those numbers, but you can also consider them in pairs. Notice that $1 + 10$, $2 + 9$, $3 + 8$, and so on, sum up to 11. Since there are 10 numbers, there are 5 pairs of numbers that sum to 11. $5 \times 11 =$ **55**.

2. In how many ways can 57 be written as the sum of two prime numbers?

Solution: 0
We know that the sum of two odd and two even numbers is even, and the sum of an odd and an even number is odd. Since 57 is odd, it cannot be the sum of two odd primes, so one of the prime numbers must be even. 2 is the only even prime number, but the other number would be 55. 55 is not prime because it has prime factors 5 and 11. Thus, there are **0** ways for 57 to be written as the sum of two prime numbers.

3. A rectangle has a width of 3. If the length is three times the width, calculate the perimeter of the rectangle.

Solution: 24
The length is $3 \times 3 = 9$. To calculate the perimeter, we multiply the sum of the length and width, $3 + 9 = 12$, by 2, to achieve **24**.

4. 4 people share 24 slices of pizza. If person A eats the same number of slices as persons B, C, and D combined, and if $B = C = D$, how many slices did person B eat?

Solution: 4

Let's assign the variable x to the number of slices person B ate. Notice that x is also the number of slices persons C and D each ate. Thus, person A ate $3x$ number of slices.

$$3x + x + x + x = 24$$

$$6x = 24$$

$$x = \mathbf{4}$$

5. Find the value of x if $4x + 28 = 12$.

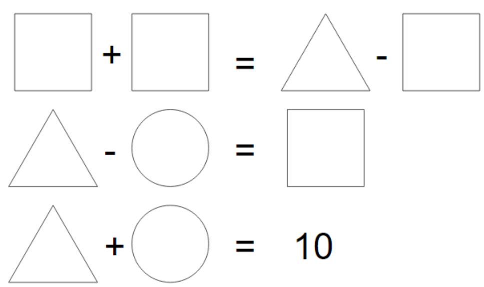
Solution: -4
Subtract 28 from both sides.

$$4x = -16$$

Divide both sides by 4.

$$x = \mathbf{-4}$$

6. Different shapes represent different positive integers. Find the sum of the square and the triangle.



Solution: 8
From the first equation, we have that 3 squares is equal to 1 triangle. Substituting that into the second equation, we have that 1 circle is equivalent to 2 squares. From the first two equations, we can replace both the triangle and the circle in the third equation with squares. 5 squares is equal to 10, so 1 square is 2. 1 triangle is 6. The sum of a square and a triangle is **8**.

7. A triangle has a base of 6. If the height is half of the base, what is the area of the triangle?

Solution: 9
The height is $\frac{1}{2} \times 6 = 3$. The area of the triangle is $\frac{1}{2} \times 3 \times 6 = \mathbf{9}$.

8. How many positive integers less than or equal to 60 are multiples of 3 or 4?

Solution: 30
There are $60 \div 3 = 20$ multiples of 3 less than or equal to 60. There are $60 \div 4 = 15$ multiples of 4 less than or equal to 60.However, we can't just add the two numbers since the numbers that are multiples of both 3 and 4 are counted twice; they are multiples of 12. There are $60 \div 12 = 5$ multiples of 12. Now we can add our first two results and subtract the numbers counted twice. $20 + 15 - 5 = \mathbf{30}$.

9. How many different ways can 16 be written as a sum of 4 different positive integers?

Solution: 9
We will first consider the smallest possible sum of three different positive integers, which is $1+2+3=6$. Using that, we can consider the problem as finding two different positive integers that sum to 16, and express one of them as a sum of 3 different integers if possible.Note that each of the 3 different integers cannot be equal to or exceed the fourth number, or you will get repeats.

$$
\begin{aligned}
16 &= 6 + 10 \\
&= 1 + 2 + 3 + 10 \\
16 &= 7 + 9 \\
&= 1 + 2 + 4 + 9 \\
16 &= 8 + 8 \\
&= 1 + 2 + 5 + 8 \\
&= 1 + 3 + 4 + 8 \\
16 &= 9 + 7 \\
&= 1 + 2 + 6 + 7 \\
&= 1 + 3 + 5 + 7 \\
&= 2 + 3 + 4 + 7 \\
16 &= 10 + 6 \\
&= 1 + 4 + 5 + 6 \\
&= 2 + 3 + 5 + 6
\end{aligned}
$$

And the list terminates here. There are **9** ways to express 16 as the sum of 4 different positive integers.

10. A square and a regular hexagon have the same perimeter. What is the ratio of the area of the hexagon to the area of the square?

Solution: $\mathbf{\frac{2\sqrt{3}}{3}}$
Since a square has 4 sides and a hexagon has 6, let's say that the perimeter is 12 as it is the least common multiple of 4 and 6. Then, the square has sides

of length 3 and an area of $3^2 = 9$. The hexagon has sides of length 2. We'll calculate its area by considering it as 6 equilateral triangles with side length 2. The formula for calculating the area of an equilateral triangle is $\frac{side^2\sqrt{3}}{4}$. The area of the hexagon is $6 \times \frac{2^2\sqrt{3}}{4} = 6\sqrt{3}$. The ratio of their areas is $\frac{6\sqrt{3}}{9} = \mathbf{\frac{2\sqrt{3}}{3}}$.

11. Billy can finish a job in 3 hours. Cristiano can finish the same job in 2 hours. How much time, in minutes, would it take them to finish the job if they worked together?

Solution: 72
Let's consider the amount of work in the job as the number 1. The speed of Billy is $\frac{1}{3}$, and the speed of Cristiano is $\frac{1}{2}$. The sum of their speeds is $\frac{5}{6}$. Now lets divide 1 by $\frac{5}{6}$ to get the time needed to finish the job when they work together, which is $\frac{6}{5}$ hours. That is equivalent to $\frac{6}{5} \times 60 = \mathbf{72}$ minutes.

12. Among all the kangaroos, the lightest two, who have the same weight, share $\frac{1}{4}$ of the total weight of the kangaroos. The heaviest three, who have the same weight, share $\frac{3}{5}$ of the total weight. How many kangaroos are there in total? (written by Tony Liu)

Solution: 6
Subtract $\frac{1}{4}$ and $\frac{3}{5}$ from 1 to get $\frac{3}{20}$, which represents the weight of the remaining kangaroos. The two lightest kangaroos each take up $\frac{1}{4} \div 2 = \frac{1}{8}$ of the total weight. The three heaviest kangaroos each take up $\frac{3}{5} \div 3 = \frac{1}{5}$ of the total weight. Now, it's easiest to consider the total weight of the kangaroos as 40, the LCM of 8, 5, and 20. Each of the lightest kangaroos has a weight of 5. Each of the heaviest kangaroos has a weight of 8. The remaining kangaroo(s) have the weight of 6. Notice that can only be the weight of one kangaroo, since there is no way to split 6 into two or more weights that are both greater than 5. So there are a total of $2 + 3 + 1 = \mathbf{6}$ kangaroos.

13. A circle is inscribed in a $3 - 4 - 5$ triangle. Find the area of the circle, expressed in terms of π.

Solution: π
The radius of the inscribed circle is equal to the area of the triangle divided by the triangle's semiperimeter. The area of the triangle is $\frac{1}{2} \times 3 \times 4 = 6$. The semiperimeter is $(3 + 4 + 5) \div 2 = 6$. The radius of the inscribed circle is $6 \div 6 = 1$. The area of the circle is $1^2\pi = \pi$.

14. 27 unit cubes are arranged into a $3 \times 3 \times 3$ cube. All the faces of the larger cube are painted blue. When randomly choosing a small cube, what is the probability of selecting one with two faces painted?

Solution: $\mathbf{\frac{4}{9}}$

Out of all 27 cubes, only one (the one in the middle) has no faces painted. The other ones have either 1, 2, or 3 faces painted. Recall that a cube has 8 vertices, 6 faces, and 12 edges. The 8 vertex cubes have 3 face painted. The 6 cubes in the middle of each shown face have 1 face painted. The remaining 12 cubes, those in the middle of each edge, have 2 faces painted. The probability of selecting one of them is $\frac{12}{27} = \frac{\mathbf{4}}{\mathbf{9}}$.

15. The sum of the ages of Bob's 4 kids is K. K is also equal to Bob's age. X years later, the sum of the kids' ages is 3 times Bob's age. What is $\frac{X}{K}$?

Solution: 2
X years later, the sum of Bob's 4 kids will be $K + 4X$ while Bob's age will be $K + X$.

$$\begin{aligned} K + 4X &= 3(K + X) \\ K + 4X &= 3K + 3X \\ X &= 2K \\ \frac{X}{K} &= \mathbf{2} \end{aligned}$$

16. There is a total of 25 humans and dogs in a neighborhood. Assume every human has 2 legs, and every dog has 4. If there are 60 legs in total, what is the positive difference between the numbers of humans and dogs?

Solution: 15
Let's set up a system of equations with the number of humans being x and the number of dogs being y.

$$x + y = 25$$

$$2x + 4y = 60$$

We can multiply the first equation by 2 to get $2x + 2y = 50$. Subtracting that from the second equations gets us $2y = 10$, so $y = 5$. We can find x from the first equation, $x = 20$. The answer is $20 - 5 = \mathbf{15}$.

17. The number $\underline{A}\ \underline{3}\ \underline{7}\ \underline{6}\ \underline{B}\ \underline{0}$ is divisible by 99. Find all possible ordered pairs (A, B).

Solution: $(\mathbf{1}, \mathbf{1})$, $(\mathbf{2}, \mathbf{0})$
We need to know some divisibility rules. If a number is divisible by 99, then it must be divisible by both 9 and 11. A number is divisible by 9 if all of its digits sum up to a multiple of 9. A number is divisible by 11 if the alternating sum of its digits from left to right is divisible by 11. For example, take 135212, $1 - 3 + 5 - 2 + 1 - 2 = 0$, and 0 is a multiple of 11, so 135212 is divisible by 11. From the divisibility rules, we know that $16 + A + B$ is a multiple of 9, and $A - 3 + 7 - 6 + B$ or $A + B - 2$ is a multiple of 11. The multiple of 9 closest

to 16 is 18, so we get two ordered pairs, $(1, 1)$ and $(2, 0)$. The next multiple of 9 is 27, so $A + B$ must be 11. However $11 - 2$ is not a multiple of 11, so 27 is out.The next multiple of 9 is 36, so $A + B$ must be 20, but 20 cannot be a sum of two one-digit integers, so this option is out too. The only ordered pairs we have are $(\mathbf{1}, \mathbf{1})$ and $(\mathbf{2}, \mathbf{0})$.

18. Let the mixed fraction $A\frac{B}{C}$ represent the value of $\frac{2+4+6+...4036}{1+3+5+...2017}$. Find the sum $A + B + C$.

Solution: 1015
Lets take a look at the numerator. Taking out a 2,

$$2(1 + 2 + 3 + ...2018) = 2 \cdot \frac{(2018)(2039)}{2} = 2018 \cdot 2019$$

The denominator is the sum of odd numbers, starting with 1. If there are n terms, the sum of the first n odd terms is n^2. There are $\lfloor \frac{2017}{2} \rfloor + 1$ terms, or 1009 terms, so the denominator is 1009^2.

Putting these together,

$$\frac{2 + 4 + 6 + ...4036}{1 + 3 + 5 + ...2017} = \frac{2018 \cdot 2019}{1009^2} = \frac{2(1009) \cdot 2019}{1009^2} = \frac{2 \cdot 2019}{1009} = \frac{4038}{1009}$$

1009 can go into 4038 4 times. So, the mixed fraction is $4\frac{2}{1009}$. Since the question asks for the sume of $A + B + C$, the answer is $4 + 2 + 1009 = \mathbf{1015}$.

19. The angles in a triangle have the ratio $2 : 3 : 4$. What is the angle measure of the largest angle?

Solution: 80°
We know that the sum of the angles in a triangle is 180°. From the given information, let's call the three angles $2x$, $3x$, and $4x$.

$$2x + 3x + 4x = 180$$

$$9x = 180$$

$$x = 20$$

The largest angle is $4x$, so it is equal to **80°**.

20. A group of teenagers and adults are discussing how to take over the world. Initially, 25% of the group are teenagers. After 3 adults and 2 teenagers leave, 20% of the group are teenagers. How many adults were there initially?

Solution: 15

Let's call the initial number of teenagers T and the initial total number of people in the group G. We can proceed to set up a system of equations.

$$T = \frac{G}{4}$$

$$T - 2 = \frac{G-5}{5}$$

Substituting the first into the second and multiplying both sides by 20, we have

$$5G - 40 = 4G - 20$$

$$G = 20$$

We now know there are 20 people in the initial group, and 5 teenagers. Thus, there were **15** adults initially.

21. Four integers a, b, c, d, not necessarily different, are chosen randomly between 1 and 36, inclusive. What is the probability that $ab - cd$ is odd?

Solution: $\mathbf{\frac{3}{8}}$
For $ab - cd$ to be odd, there must be one even and one odd between ab and cd. First consider that ab is even and cd is odd. For ab to be even, at least one of a and b has to be even. For cd to be odd, both c and d have to be odd. We have three cases. $ab - cd$ can be $EVEN \times ODD - ODD \times ODD$, $Even \times Even - ODD \times ODD$, or $ODD \times EVEN - ODD \times ODD$. Each case has a probability $(\frac{1}{2})^4 = \frac{1}{16}$, because half of the numbers $1 - 36$ is even and half is odd. Since there are three cases, the combined probability is $\frac{3}{16}$.
Now consider that ab is odd and cd is even. This has the same probability as before, which is $\frac{3}{16}$.
The final answer is $\frac{3}{16} \times 2 = \mathbf{\frac{3}{8}}$.

22. In the addition equation below, different letters represent distinct positive digits (1-9). Given that $D = 1$ and $M = 8$, find the four-digit number $\underline{MATH}$[1]

$$\begin{array}{r} \text{H A R D} \\ + \ \text{G O O D} \\ \hline \text{M A T H} \end{array}$$

[1] There are multiple solutions; any that work are acceptable.

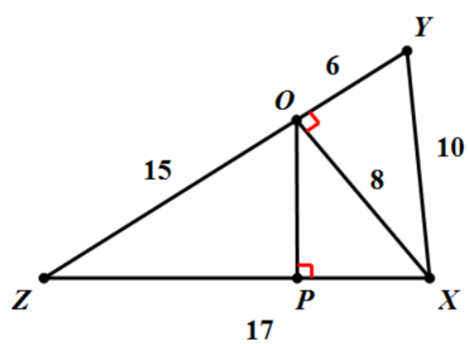
Solution: 8632, 8732, 8362, or 8462
Knowing that $D = 1$, we can immediately get that $H = 2$. Since $A + O = A$ while R and T must be different values, $O = 9$. $A + O$ exceeds 10, so we have to carry 1 into $H + G$. $H + G + 1 = M = 8$, since $H = 2$, $G = 5$.This leaves $3, 4, 6, 7$ for A, R, T. R cannot be 3 or 6, because then $T = 2$ or $T = 5$. Trying $R = 4$ and $R = 7$ will yield you the results $\mathbf{8632}, \mathbf{8732}, \mathbf{8362}, \mathbf{8462}$.

23. A number is called a "good number" if all of its factors excluding itself add up to the number itself. For example, 6 is a "good number" since $6 = 1 + 2 + 3$. Find the next "good number" after 6.

Solution: 28
The simplest way is trial-and-error. Skipping prime numbers, you will find the next good number is 28. $28 = 1 + 2 + 4 + 7 + 14$. The proper term of numbers like 6 and 28 is "Perfect Number". For more information, you can visit `https://en.wikipedia.org/wiki/Perfect_number`.

24. In triangle XYZ, the height from X is drawn, hitting segment YZ at a point O. Then, a height of triangle XOZ from O is drawn, hitting side XZ at a point P. Given that $XY = 10$, $OY = 6$, $XZ = 17$, find exactly the length of OP.

Solution: $\frac{120}{17}$



From the given information, we can find $OX = 8$ by the Pythagorean Triple $6-8-10$. Then we can find $OZ = 15$ from the Pythagorean Triple $8-15-17$. ΔOPX is similar to ΔZOX. We can then set up a proportion to find OP.

$$\frac{OP}{OX} = \frac{ZO}{ZX}$$

$$\frac{OP}{8} = \frac{15}{17}$$

$$OP = \frac{\mathbf{120}}{\mathbf{17}}$$

25. Determine exactly $2x_1 + x_3$ if x_1, x_2, x_3, x_4, and x_5 satisfy the system of equations below.

$$3x_1 + x_2 + x_3 + x_4 = 8$$
$$x_1 + 3x_2 + x_3 + x_4 = 21$$
$$x_1 + x_2 + 3x_3 + x_4 = 36$$
$$x_1 + x_2 + x_3 + 3x_4 = 54$$

Solution: $\mathbf{\frac{149}{12}}$
Adding the four equations, we get:

$$6x_1 + 6x_2 + 6x_3 + 6x_4 = 119$$
$$x_1 + x_2 + x_3 + x_4 = \frac{119}{6}$$

If we subtract the first equation and the third equation from the one we just found, we can find both x_1 and x_3

$$2x_1 = 8 - \frac{119}{6} = -\frac{71}{6}$$
$$2x_3 = 36 - \frac{119}{6} = \frac{97}{6}$$

So $3x_3 = \frac{3}{2} \cdot \frac{97}{6} = \frac{97}{4}$ Adding these two answers,

$$-\frac{71}{6} + \frac{97}{4} = -\frac{142}{12} + \frac{291}{12} = \frac{149}{12}$$

26. Lebron swims and runs at constant rates. He runs three times as fast as he swims. In a race, Lebron completes the swimming part in 30 minutes and the running part in 2 hours. Find exactly the ratio of the distance he travels swimming to the distance he travels running.

Solution: $\mathbf{\frac{1}{12}}$
Let's call his speed of swimming s, then his speed of running is $3s$. We can use the formula $distance = time \times speed$ to find the distances he travels. $30 = \frac{1}{2}hour$, the distance he travels swimming is $\frac{1}{2}s$. The distance he travels running is $2 \times 3s = 6s$. The ratio is $\frac{\frac{s}{2}}{6s} = \mathbf{\frac{1}{12}}$.

27. A certain unfair six-sided die with faces numbered $1-6$ has the property that the probability of rolling a number is proportional to the number's value. When two such dies are rolled, what is the probability that the sum of the two numbers is 6?

Solution: $\mathbf{\frac{31}{441}}$
First we will find the probability of attaining each number. The denominator will be $1 + 2 + 3 + 4 + 5 + 6 = 21$.The probability of attaining 1 will be $\frac{1}{21}$, 2 will be $\frac{2}{21}$, and so on.
When two dies are rolled, the pairs that sum to 6 are $(1, 5)$, $(2, 4)$, and $(3, 3)$. For the first pair, the probability is $2 \times \frac{1}{21} \times 521 = \frac{10}{441}$ since the numbers can appear on different dies. The probability for the second pair is $2 \times \frac{2}{21} \times \frac{3}{21} = \frac{12}{441}$. The probability for the last pair is $\frac{3}{21} \times \frac{3}{21} = \frac{9}{441}$. Adding all those fractions yields the answer of $\mathbf{\frac{31}{441}}$.

28. When a number is divided by 3, the remainder is 2. When the same number is divided by 5, the remainder is 4. When the same number is divided by 7, the remainder is 1. Find the second least positive number that satisfies these conditions.

Solution: 134
You can use trial-and-error to find the smallest positive number that satisfies these conditions, 29. Then you just need to add to it the LCM of 3, 5, and 7, which is $3 \times 5 \times 7 = 105$. $29 + 105 = \mathbf{134}$. However, if you want to know how to solve these types of problems no matter how large the numbers, please read on. This problem deals with modular arithmetic. Call the number we're solving for x. In a modular arithmetic, $a \equiv b \mod m$ means that $a - b$ is divisible by m, or that b is the remainder when a is divided by m. From the given information, we can set up the equations

$$x \equiv 2 \mod 3$$

$$x \equiv 4 \mod 5$$

$$x \equiv 1 \mod 7$$

The Chinese Remainder Theorem guarantees that there's a solution to this system of congruence equations since $3, 5, 7$ are relatively prime. You can learn more about the Chinese Remainder Theorem here: `https://en.wikipedia.org/wiki/Chinese_remainder_theorem`
From the first equation, we can express x as $3r + 2$ for some integer r. Substituting that into the second equation, we have

$$\begin{aligned} 3r + 2 &\equiv 4 \mod 5 \\ 3r &\equiv 2 \mod 5 \\ r &\equiv 4 \mod 5 \end{aligned}$$

Since $3r \equiv 2 \mod 5$, $3r$ has a remainder of 2 when divided by 5, and the smallest possible value satisfying this condition is 12, so $r \equiv 4 \mod 5$. Now, we can express r as $5s + 4$ for some integer s. We will express x in terms of s

by substituting $5s+4$ into r. $x = 3(5s+4)+2 = 15s+14$. We will use this expression of x in the third equation.

$$\begin{aligned} 15s + 14 &\equiv 1 \mod 7 \\ 15s &\equiv 1 \mod 7 \\ s &\equiv 1 \mod 7 \end{aligned}$$

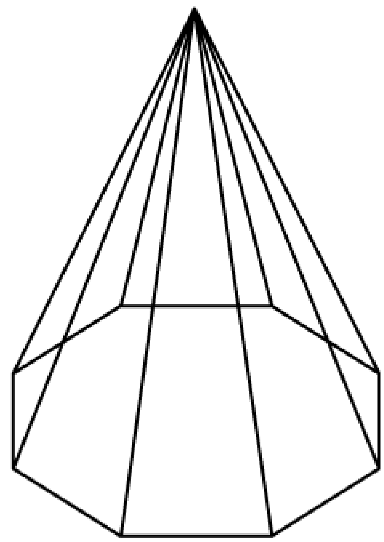
Because 14 is $0 \mod 7$, so we can conclude that $15s \equiv 1 \mod 7$. 15 is the smallest multiple of 15 that satisfies $1 \mod 7$, so $s \equiv 1 \mod 7$. Now express s as $7k+1$ for some integer k. We will substitute this into $15s+14$. $x = 15(7k+1)+14 = 105k+29$. $105k+29$ for any integer k is actually the expression used to find all possible x's, and there is an infinite number of solutions. The smallest positive x is when $k=0$, so the second smallest positive x is when $k=1$, and $x = \mathbf{134}$.

29. Richard has four identical fair coins with a distinguishable head and tail. What is the probability that at least two of the coins land heads?

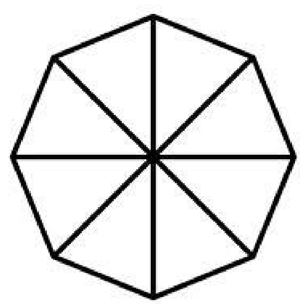
Solution: $\mathbf{\frac{11}{16}}$
We can use complementary counting for this; at least two coins landing heads means that we don't want zero or one coin landing heads. If zero coins land heads, then all four land tails, which has a probability of $(\frac{1}{2})^4 = \frac{1}{16}$. The probability of one coin landing heads is $4 \times (\frac{1}{2}) \times (\frac{1}{2})^3 = \frac{1}{4}$. We multiply by 4, because there are 4 coins that are candidates for the one coin that land heads. Now we subtract $\frac{1}{16}$ and $\frac{1}{4}$ from 1 to get the answer of $\mathbf{\frac{11}{16}}$.

30. A right regular octagonal pyramid $ABCDEFGHI$ with A as the vertex has a base with length 4. Given that $AB = 10$, find exactly the volume of the pyramid.

Solution: $\mathbf{\dfrac{32+32\sqrt{2}}{3} \cdot \sqrt{21-2\sqrt{2}}}$



The area of a pyramid, no matter the base, is $\frac{1}{3}bh$, where b is the area of the base and h is the height of the pyramid. The area of the base can be found by finding the area of the octagon. The height can be calculated using the Pythagorean Theorem, using the slant height and the distance from one base vertex to the center of the octagon (the lines as shown below).



Using Law of Cosines on a triangle in the regular octagon,

$$4^2 = x^2 + x^2 - 2x^2 \cos 45$$

, where x is a side of the isoceles triangle. Solving for x,

$$16 = 2x^2(1 - \cos 45)$$

$$\frac{8}{1-\cos 45} = x^2$$

$$\frac{8}{1-\frac{\sqrt{2}}{2}} = x^2$$

$$\frac{8}{\frac{2-\sqrt{2}}{2}} = x^2$$

$$\frac{16}{2-\sqrt{2}} = x^2$$

$$\frac{16}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}} = x^2$$

$$8(2+\sqrt{2}) = x^2$$

This is one of the sides of the right triangle we need to find the height of this pyramid. The hypotenuse is given as 10.
SO,

$$10^2 = x^2 + h^2$$

$$100 = 8(2+\sqrt{2}) + h^2$$

Solving and simplifying,

$$h = 2\sqrt{21-2\sqrt{2}}$$

All that is left to find the volume of the figure is to calculate the area of the octagon. We can find the area of any one of the eight congruent triangles by using $A = \frac{1}{2}ab\sin C$, where A is the area, a and b are side lengths, and C is the angle between sides a and b.
In this case, $a = b$ as every triangle drawn is isosceles (in a regular octagon). Thus, $a = b = x$, so we just want $\frac{1}{2}x^2 \sin 45 = 4\sqrt{2}+4$. As there are 8 triangles, the area of the octagon is $32\sqrt{2} + 32$. Finally, the volume of this figure is just $\frac{1}{3}bh = \frac{32+32\sqrt{2}}{3} \cdot \sqrt{21-2\sqrt{2}}$.